Geometric Satake Correspondence

Max Steinberg

22 November 2023

1 Review

We start by reviewing a couple basic notions we will need later.

Definition 1 (Langlands dual)**.** *Let G be a reductive group over a field k with torus T and root system $(X^\bullet(T), X_\bullet(T), \{\alpha\}, \{\alpha^\vee\})$. The **Langlands dual of G over** k, denoted $G_k^\vee$, or simply $G^\vee$ when k is clear, is a reductive group (under the Chevalley correspondence) with root system $(X_\bullet(T), X^\bullet(T), \{\alpha^\vee\}, \{\alpha\})$ (swapping roots and coroots).*

Definition 2 (perverse sheaf)**.** *We will not define perverse sheaves in detail. In essence, a "perverse sheaf" on a variety X is some complex of sheaves that lives inside an abelian category* $\mathrm{Perv}(X) \subset D_c^b(X)$*. We will not define* $\mathrm{Perv}(X)$ *(other than as some mumbo-jumbo "heart of the perverse t-structure").*

Definition 3 (IC sheaf)**.** *An IC-sheaf is a specific perverse sheaf. We will not define what it is!*

We can also consider perverse sheaves in the heart of the perverse t-structure of G-equivariant (constructible) sheaves when G acts on X. In this context, we write $\mathrm{Perv}_G(X)$ to denote this abelian category. Importantly, perverse sheaves admit all the natural functors sheaves admit, but as derived functors. In particular, we will be looking at the stalks of perverse sheaves at points, which will be a *complex* of vector spaces, rather than a single vector space.

2 Geometric Satake Correspondence

Today, we will go over all of the necessary ingredients in the Geometric Satake Correspondence that we will need next time.

Theorem 1 (Geometric Satake Correspondence)**.** *Let G be a complex reductive group, k be a Noetherian ring of finite global dimension, and denote $G_k^\vee$ as the Langlands dual of G over k. We define the **Satake category** of G over k to be*

$$\mathrm{Sat}_{G,k} := \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, k)$$

Then, we have the following commutative diagramme of symmetric monoidal categories over k-modules:

$$\begin{array}{ccc} \mathrm{Sat}_{G,k} & \xrightarrow{\cong} & \mathrm{Rep}_k(G^\vee) \\ & {\scriptstyle H^*(-)} \searrow \quad \swarrow {\scriptstyle forget} & \\ & Mod_k & \end{array}$$

Statement due to [Fan]. $H^*(-)$ is hypercohomology, which we will not discuss. This offers a brand-new (to us) construction of the Langlands dual group that has nothing to do with root systems. Logan will be able to offer more justification for this theorem (or possibly even a sketch of a proof) next time. A posteriori, this equivalence can be restated as the following claim: $\mathrm{Sat}_{G,k}$ is a Tannakian category equivalent to the category of representations of $G^\vee$ over k. The proof of the Geometric Satake Correspondence uses the Tannakian formalism to show that $\mathrm{Sat}_{G,k}$ is a Tannakian category, and in particular, dual to $G^\vee$. We will outline a sketch of this process in this talk, and Logan will (probably) fill in the details next time.

3 Stratification of the Affine Grassmannian

For this section, we fix a complex reductive group G and write Gr_G as its affine Grassmannian. We write $K = \mathbb{C}((t))$ and $O = \mathbb{C}[[t]]$. We start by describing several decompositions G, following the Bruhat decomposition in the classical setting:

Theorem 2 ([Fan] 2.4)**.** *We have the following decompositions:*

1. $G(\mathbb{C}) = BWB = \bigsqcup_{w\in W} B(\mathbb{C})wB(\mathbb{C})$ *(Bruhat)*
2. $G(K) = IW^{aff}I = \bigsqcup_{w\in W^{aff}} I(K)wI(K)$ *(Iwahori-Bruhat)*
3. $G(K) = G(O)X_\bullet(T)^+G(O) = \bigsqcup_{\lambda\in X_\bullet(T)^+} G(O)t^\lambda G(O)$ *(Cartan)*
4. $G(K) = G(\mathbb{C}[t^{-1}])X_\bullet(T)^+G(O) = \bigsqcup_{\lambda\in X_\bullet(T)^+} G(\mathbb{C}[t^{-1}])t^\lambda G(O)$ *(Birkhoff)*
5. $G(K) = B(K)G(O)$ *(Iwasawa)*

These decompositions relate directly to the decomposition Luna gave in her talk:

$$\mathrm{GL}_n(K) = \bigsqcup \mathrm{GL}_n(O)t^\gamma \,\mathrm{GL}_n(O),$$

where γ is a dominant cocharacter of the torus. Under the action of the Weyl group, which is S_n in this case, every cocharacter of the torus is in an orbit of a dominant cocharacter: a cocharacter is a map $\mathbb{G}_m \to T$, ie. a map $\mathbb{G}_m \to \mathbb{G}_m^n$, ie. a map $\mathbb{Z} \to \mathbb{Z}^n$, a choice of n integers. Up to symmetry, we can make the first integer the largest, which makes γ dominant.

Again following the classical case, we consider Schubert cells in LG/L^+G, just as we do for left B-orbits in G/B. For $g \in LG$, we write $[g]$ to denote its L^+G-orbit in $\mathrm{Gr}_G = LG/L^+G$. Using our Cartan decomposition, we can write $G(K) = G(O)X_\bullet(T)^+G(O)$, and we can use this to build a decomposition (which we will shortly show is a stratification) of Gr_G. Let $\lambda \in X_\bullet(T)^+$ and write Gr_λ to denote the L^+G-orbit containing $[t^\lambda]$. We can see that $\mathrm{Gr}_G = \sqcup \mathrm{Gr}_\lambda$ (ie. we recover every L^+G orbit) by the Birkhoff decomposition, but it is not clear that these pieces Gr_λ form a stratification. We will show this now.

Proposition 1 ([Fan] 2.5)**.** *We have the following:*

1. $\overline{\mathrm{Gr}_\lambda} = \mathrm{Gr}_{\leq\lambda} := \bigsqcup_{\mu\leq\lambda,\mu\in X_\bullet(T)^+} \mathrm{Gr}_\mu$
2. Gr_λ *is an affine bundle over the partial flag manifold* G/P_λ *of* $\dim(\mathrm{Gr}_\lambda) = \langle 2\rho, \lambda\rangle$ *(where* P_λ *is the parabolic subgroup of* G *spanned by the negative root perpendicular to* λ*)*

Thus, $\{\mathrm{Gr}_\lambda\}$ in fact stratify Gr_G. This is where we get our first inkling of a connection to $G^\vee$: recall that, by definition, $X_\bullet(T) = X^\bullet(T^\vee)$. So

$$\mathrm{Gr}_G = \bigsqcup_{\lambda\in X_\bullet(T)^+} \mathrm{Gr}_\lambda = \bigsqcup_{\lambda\in X^\bullet(T^\vee)^+} \mathrm{Gr}_\lambda$$

So in a certain sense, the affine Grassmannian of G is controlled by the representations of $G^\vee$! There is a lot of technical machinery needed to establish the details, but at a high level, we have perverse sheaves called the "IC sheaves" and when we take the stalks of IC sheaves at any point $[\lambda]$, we have a connection to representations of $G^\vee$. Recall that the stalk of a perverse sheaf is a complex of vector spaces. If we denote $n(V) = \sum \dim(V_i)$ when V is a complex of vector spaces, then we find that

$$n(IC_{[\lambda]}) = \dim G^\vee_\lambda$$

where the right-hand side denotes the dimension of the λ-weight space corresponding to the dominant weight λ of $G^\vee$. Let's look at an example.

Example 1. *Let* $G = \mathrm{SL}_2$ *over* $\mathbb{C}$*. The affine Grassmannian of* SL_2 *over* $\mathbb{C}$ *can be identified with* $\mathbb{A}^0 \sqcup \mathbb{A}^2 \sqcup \dots$*, as a variety.* $\mathbb{A}^{2i}$ *corresponds to the highest-weight representation of* PGL_2 *with highest weight* $2i$*. So we have a bijection between cells of* $\mathrm{Gr}_{\mathrm{SL}_2}$ *and irreps of* PGL_2*!*

4 Tannakian Formalism

Definition 4 (Tannakian category)**.** *Let* K *be a field and* C *an abelian rigid tensor category such that* $\mathrm{End}(1) \cong K$*. We say* C *is* ***Tannakian over*** K*, or* ***Tannakian*** *when* K *is clear, if there is some field extension* L *of* K *such that there is some* K*-linear, exact, faithful tensor functor* $F : C \to L\text{-}\mathsf{Vect}$*. We call* F *the* ***fibre functor*** *of* C*.*

Tannakian categories are a natural extension of the category of representations of a group G over a field K. For every field K and every algebraic group G, $\mathrm{Rep}_K(G)$ is a Tannakian category, with F the forgetful functor to K-vector spaces. When $L = K$, we say C is **neutral**. Thus, every $\mathrm{Rep}_K(G)$ is a neutral Tannakian category. Similarly, $\mathrm{Rep}_L(G)$ is Tannakian over F for any L/F in the same way, but not neutral. Tannakian categories are one half of **Tannaka-Krein Duality**: the duality between G and $\mathrm{Rep}_K(G)$. This duality is integral to the proof of Geometric Satake: we first prove that a certain category is Tannakian and satisfies the conditions for Tannaka-Krein duality, and thus, it must be the category $\mathrm{Rep}_K(H)$ for some group H. Then, we show that H is in fact $G^\vee$, the Langlands dual of G.

Tannaka-Krein duality was built in two parts: Tannaka's Reconstruction Theorem and Krein's Theorem. Tannaka's Reconstruction Theorem allows us to extract G from $\mathrm{Rep}(G)$, and Krein's Theorem gives sufficient conditions for a Tannakian category to be $\mathrm{Rep}(G)$ for some G. These two pieces together allow us to prove that the Satake category of G over Gr_G is equivalent to $\mathrm{Rep}_k(G^\vee)$. So let us first look at Tannaka's Reconstruction Theorem.

Theorem 3 (Tannaka's Reconstruction Theorem [Pie] 2.11)**.** *Let C be a neutral Tannakian category over k with fibre functor F. Then,*

1. *$\underline{\mathrm{Aut}}^\otimes(F)$ is represented by an algebraic group G as a functor of k-algebras*
2. *$C \to \mathrm{Rep}_k(G)$ given by F is an equivalence of tensor categories*

The full statement for potentially non-neutral Tannakian categories requires more technology such as gerbes. The following theorem is incorrect. It was given as written in the talk, but as written it is clearly incorrect. A corrected statement can be found in Milne-Deligne's notes, but for posterity, the notes for the talk have not been modified.

Theorem 4 (Krein's Theorem)**.** *Let D be a category of finite-dimensional vector spaces (ie. a subcategory of Vect_k), under the tensor product. If the following conditions hold:*

1. *there is some $I \in D$ with $I \otimes A \cong A$*
2. *every object $A \in D$ is a sum of "minimal objects"*
3. *for any two minimal objects V, W, $\mathrm{Hom}(V, W)$ is either one-dimensional or zero*

then D is a Tannakian category equivalent to $\mathrm{Rep}_k(G)$ where G is the group of representations of D.

4.1 Tannakian Reconstruction

We will now describe Tannakian reconstruction. We start with some observations on module categories:

Theorem 5. *Let R be a k-algebra. Then $F : R\text{-}\mathsf{Mod} \to k\text{-}\mathsf{Mod}$ is the forgetful functor, and R can be identified with the opposite ring of* $\underline{\mathrm{End}}(F)$.

Theorem 6. *Let A be an abelian category and $P \in A$. Then $\mathrm{Hom}(P, -) : A \to \mathrm{End}(P)^{op}\text{-}\mathsf{Mod}$ is an equivalence if and only if P is a compact projective generator.*

For simplicity we will only consider the case k is a field. Let D be a rigid abelian symmetric monoidal category and $F : D \to k\text{-}\mathsf{Mod}$ an exact faithful monoidal functor. Since the category k-Mod is only finite-dimensional vector spaces, it is impossible to find a single generator. So instead, consider $X \in D$ and let $\langle X\rangle$ be the full subcategory of subquotients of $X^{\oplus n}$. If $F_X : \langle X\rangle \to k\text{-}\mathsf{Mod}$ is representable, then it will be the forgetful functor a module category by the previous theorems. So we will construct some $P_X \in \langle X\rangle$ a compact projective generator. The following section is copied from [Fan] (mostly because I don't understand it well enough to rewrite it).

Let $\mathcal{C} = \langle X\rangle$. Consider functor category $\mathrm{Hom}_{Cat}(\mathcal{C}, Mod_{\mathrm{k}})$, with typical objects constant functor V for $V \in \mathrm{Mod}_{\mathrm{k}}$ and $Y \in \mathcal{C}$ via the Yoneda lemma. Define $\mathrm{Hom}(V, Y)(Z) = \mathrm{Hom}_{Mod_k}(V, \mathrm{Hom}_{\mathcal{C}}(Y, Z)), Z \in \mathcal{C}$, which is representable by choosing a basis. Similarly we can define a tensor product: define

$$P_X = \bigcap \left(\mathrm{Hom}(F(X), X) \cap \ker\left(\mathrm{Hom}\left(F\left(X^{\oplus n}\right), X^{\oplus n}\right) \to Hom\left(F(Y), X^{\oplus n}/Y\right)\right)\right),$$

where we intersect over $\{n \geq 0, Y \subset X^{\oplus n}\}$ and Hom(F(X), X) is diagonally embeded in $\mathrm{Hom}\left(F\left(X^{\oplus n}\right), X^{\oplus n}\right)$. We apply F and calculate: $A_X := F(P_X)$

$$= \bigcap_{n\geq 0, Y\subset X\oplus n} \left(\mathrm{End}(F(X)) \cap \ker\left(\mathrm{End}\left(F\left(X^{\oplus n}\right)\right) \to Hom\left(F(Y), F\left(X^{\oplus n}\right)/F(Y)\right)\right)\right)$$
$$= \left\{A \in \mathrm{End}(F(X)) \mid \forall Y \subset F\left(X^{\oplus n}\right), A(F(Y)) \subset F(Y) \text{ as subsets of } F\left(X^{\oplus n}\right)\right\}$$

Observing F automatically factors as $\mathcal{C} \to \mathrm{Mod}_{A_X} \to \mathrm{Mod}_{\mathrm{k}}$, we claim the first arrow is an equivalence. Note there is a homomorphism $A_X \to \mathrm{End}_{\mathcal{C}}(P_X)^{op}$ by composition so that we can define an inverse by

$$P_X \otimes_{A_X} M := \mathrm{coker}\left(P_X \otimes A_X \otimes M \rightrightarrows P_X \otimes M\right),$$

where the two arrows are two actions. We can easily check $F(P_X \otimes_{A_X} -) = Id$ and hence F is fully faithful. Lifting the first arrow of a presentation $A_X^m \to A_X^n \to M \to 0$ in Mod_{A_X}, we get F is essentially surjective.

The second step takes a limit of the local realizations above. However, module category over rings is not compatible with the limiting process we need, because the category of $\lim A_i$-modules is not equivalent to the colimit of category of A_i-modules. To make the limit process correct, we should use comodules instead of modules. So denote $B_X := A_X^\vee$, dual operator induces an equivalence $\mathrm{Comod}_{B_X} = \mathrm{Mod}_{A_X}$. For $\langle X\rangle \subset \langle X'\rangle$, there is an arrow $A_{X'} \to A_X$ by restriction, so $B_X \to B_{X'}$. We define $B = \varinjlim B_X$ and obtain the equivalence $\mathcal{C} \cong \mathrm{Comod}_B$

The third step adds monoidal structure into the picture. Now we've realized $\mathcal{C} \cong \text{Comod}_B$, a symmetric monoidal structure $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is the same as $\text{Comod}_B \times \text{Comod}_B \to \text{Comod}_B$ and then $B \times B \to B$. This makes B a commutative algebra and bialgebra, or makes $\text{Spec}\, B$ a monoidal scheme in the opposite category. We want to look for criterions to make $\text{Spec}\, B$ into a group scheme. Recall when we're constructing the dual of a vector space V with group G-action, the inverse operation in G is needed to make G acting on the correct side of $V^\vee$, the dual of V. Actually the converse holds. If "dual" formally exists inside the category (so called "Tannakian category"), we can obtain an antipode map making B a Hopf algebra and $G = \text{Spec}\, B$ a group scheme. Actually we can make it even simpler: it suffices assume dual exists for "line bundle objects" with respect to F. Assume that for any $X \in \mathcal{C}$ with $\dim_{\mathrm{k}} F(X) = 1$, there exists $X^\vee \in \mathcal{C}$ and an isomorphism $X \otimes X^\vee \cong 1$.

To show G is a group scheme, we only need to show $G(R)$ is a group for any commutative algebra R over k. We first show the R-points act by isomorphism on any representation $X \in \mathcal{C}$. If $\dim_{\mathrm{k}} F(X) = 1$, then R-point simultaneously acts on $X, X^\vee$ and trivially on 1 . The isomorphism $X \otimes X^\vee \cong 1$ in the assumption the indicates the action on $F(X) \otimes_{\mathrm{k}} R$ is invertible. For general X, we take the determinant $\Lambda^{\dim_k F(X)} X$ to reduce to dimension 1 case. Glueing finite dimensional representations together, $G(R)$ acts on B by isomorphisms, hence acts bijectively on $H_{om}(B, R) = G(R)$. This implies $G(R)$ is a group since the action coincides with the natural group action.

(end copied section). There's a lot that can be said about Tannaka-Krein duality (it was even a proposed topic for our seminar next quarter). But with these basics in place, we can start sketching the proof of the Geometric Satake Correspondence. We will introduce further theory of Tannaka-Krein duality as needed in the sketch.

5 Sketch of Proof

Now, let's give a sketch of the proof of the Geometric Satake Correspondence. This is *significantly* oversimplified and lacks many crucial details. The purpose of this sketch is just to show how the different moving pieces fit together. Next week, Logan will provide a better overview of the proof in more detail.

Proposition 2. *The Satake category of G satisfies the following:*

1. *for $A, B \in G$, $A * B \in G$ (ie. the Satake category is closed under convolution)*

2. *convolution makes $\text{Sat}_{G,k}$ a symmetric monoidal category*

3. *the forgetful functor $F : \text{Sat}_{G,k} \to X^\bullet(T)$-graded k-modules is monoidal*

In fact, F is exact and faithful.

In particular, when k is of characteristic 0, we have the following nice facts.

Lemma 1. *Let IC_λ be the IC-sheaf on* Gr_λ. *Then* $H^n(IC_\lambda) \neq 0 \implies n \equiv \dim \mathrm{Gr}_\lambda \mod 2$.

Lemma 2. *The Satake category is semisimple.*

With this functor F, we can show that $(\mathrm{Sat}_{G,k}, F)$ has the structure of a Tannakian category. In fact, $\mathrm{Sat}_{G,k}$ satisfies the criterion to apply Krein's Theorem, so is in fact a representation category. Now, all that remains is to show it is the representation category of $G^\vee$. Let $\tilde{G}$ be a reductive group obtained from Tannakian reconstruction so that $\mathrm{Sat}_{G,k} = \mathrm{Rep}_k(\tilde{G})$. We claim $\tilde{G} = G^\vee$. Specifically, we claim that $\tilde{G}$ is a split reductive group over k and that G and $\tilde{G}$ have dual root systems (again over k). We can show this in several steps:

1. $\tilde{G}$ is a connected group scheme of finite type
2. $T^\vee \subset \tilde{G}$ by the structure of the Satake category over $X^\bullet(T^\vee)$ (we have a functor $F : \mathrm{Sat}_{G,k} \to \mathsf{Mod}_k[X_\bullet(T)] \cong \mathrm{Rep}_k(T^\vee)$ via the weight structure)
3. $T^\vee$ is maximal in $\tilde{G}$
4. the set of roots within $\tilde{G}$ and $G^\vee$ are identical

and from this, we see that in fact $\tilde{G} = G^\vee$ (up to isogeny).

References

[Fan] Xingzhu Fang. *AN INTRODUCTION TO GEOMETRIC SATAKE EQUIVALENCE.*

[Pie] James Milne Pierre Deligne. *Tannakian Categories.*