Representation Theory

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August 2023

Contents

1 Introduction

These notes consist of a gentle introduction to representation theory of finite groups and quivers. The main goal of these notes is to introduce quivers, their repesentations, and the classical McKay correspondence in an intuitive and simple way, building up from the basics. Contrary to every other resource to learn quiver representations, we do not assume advanced background knowledge such as homological algebra or algebraic geometry. Instead, we introduce quivers as a natural object to consider, and examine the information we can gather through their representations. McKay Correspondence arrives as a striking result in *quaternionic geometry* that relates representations of quivers to the so-called **ADE Classification**, a fundamental result at the core of many branches of modern mathematics. ADE Classification appears in string theory, representation theory, and spherical/quaternionic geometry, and connection between these fields through ADE Classification is incredible.

In Section 2, we introduce the so-called Linear Representation Theory of Finite Groups. Finite groups are a well-behaved class of algebraic objects, and their representations exhibit many nice properties that allow us to study the structure of the groups. We will see that finite groups exhibit **complete reducibility** over $\mathbb{C}$, which means that every representation of a finite group can be built from "simple pieces."

In Section 3, we introcuce quivers and their representations. We will see how quivers relate the the ADE Classification via so-called **finite type** quivers (mirroring complete reducibility in Section 2), and later on see the ADE Classification again via McKay Correspondence.

1.1 Prerequisites

For most of these notes, knowledge of finite group theory (groups, subgroups, normal subgroups, group homomorphisms) is assumed. Each section has its own Prerequisites section that describes in more detail what is assumed for that section.

Notation

If any of the following doesn't make sense, don't worry, it will be clear when it is used.

1. All vector spaces are finite-dimensional complex vector spaces unless otherwise specified.

2. We differentiate $\vec{Q}$ and Q: $\vec{Q}$ represents a quiver and Q represents the underlying graph of a quiver (this will make sense when quivers and graphs are introduced).

2 Representations of Finite Groups

2.1 Prerequisites

1. **finite group theory**: subgroups, normal subgroups, group actions, orbit-stabiliser, etc.

2. **linear algebra**: vector spaces, bases, subspaces, linear operators, inner products, etc.

2.2 Representations

In representation theory, the central idea we study is a *representation*. Given a finite group G, instead of studying the structure of G as a group, we can investigate how G *acts* on a vector space V. This allows us to examine information about G, potentially without even knowing the full structure of G!

Definition 1 (representation)**.** *Let G be a finite group and V a vector space. We define a **representation** of G on V to be a group homomorphism $\rho : G \to \mathrm{GL}(V)$. We call this a **representation of** G **on** V.*

$\mathrm{GL}(V)$ denotes the *general linear group* on V, ie. the group of invertible linear maps $V \to V$. This can also be notated $\mathrm{Aut}(V)$, but we will use $\mathrm{GL}(V)$. If V is a vector space with dimension n, this can also be written as $\mathrm{GL}_n(\mathbb{C})$.

Definition 2 (dimension)**.** *We say the **dimension** of a representation $\rho : G \to \mathrm{GL}(V)$ is $\dim(\rho) = \dim(V)$.*

Let us look at an example in depth. Let $G = C_2 = (\{0, 1\}, +)$, ie. the group of integers with addition modulo 2 (aka the cyclic group of order 2, hence the notation C_2). We can define a representation of G on the complex vector space $\mathbb{C}$ by $\rho : G \to \mathrm{GL}(\mathbb{C})$ by $\rho(0) = 1$ and $\rho(1) = -1$. Then the dimension of ρ is 1, as $\mathbb{C}$ is a 1-dimensional complex vector space.

We can verify this is in fact a representation of G: we need to show $\rho : G \to \mathrm{GL}(\mathbb{C})$ is a group homomorphism. Recall that a group homomorphism $f : A \to B$ (with A and B groups) must satisfy

1. $f(e_A) = e_B$, where e_A is the identity in A and e_B is the identity in B

2. $f(a \cdot a') = f(a) \times f(a')$, where $\cdot$ is multiplication in A and $\times$ is multiplication in B

So let us verify. We first want to figure out exactly what $\mathrm{GL}(\mathbb{C})$ is. Since $\mathbb{C}$ is a 1-dimensional complex vector space, a linear map from $\mathbb{C}$ to $\mathbb{C}$ is a 1-by-1 matrix, ie. a scalar. We defined the general linear group to be *invertible* linear maps, which in this case is every map that is not the zero map. So $\mathrm{GL}(\mathbb{C}) = \mathbb{C}^\times$, where $\mathbb{C}^\times$ represents the group of nonzero complex numbers under *multiplication*.

So, we have a map $\rho : G \to \mathbb{C}^\times$ given by $\rho(0) = 1$ and $\rho(1) = -1$. In G, 0 is the identity, and in $\mathbb{C}^\times$, 1 is the identity. So condition (1) is satisfied. And $\rho(0+1) = \rho(1+0) = \rho(1) = -1$, which is the same as $\rho(0) \cdot \rho(1) = 1 \cdot -1 = \rho(1) \cdot \rho(0) = -1$. Further, $\rho(1+1) = \rho(0) = 1 = \rho(1) \cdot \rho(1) = -1 \cdot -1 = 1$. So we have shown that $\rho(z+z') = \rho(z)\rho(z')$ for any $z, z' \in G$. So ρ is a homomorphism from $G \to \mathrm{GL}(\mathbb{C})$, so ρ is in fact a representation of G.

Exercise 1. *Prove that the map $\rho : C_3 \to \mathrm{GL}(\mathbb{C})$ given by $\rho(n) = e^{2\pi i k/3}$ is a representation of $\mathbb{Z}/3\mathbb{Z}$.*

Definition 3 (trivial representation)**.** *Given a group G and a vector space V, the map $\rho : G \to \mathrm{GL}(V)$ given by $\rho(g) = I$ (where I is the identity matrix) is a representation of G, which we call the* ***trivial representation****.*

Every group has a trivial representation on any vector space. Representations on their own are interesting, but we can also combine them in two different ways.

2.3 Sums of Representations

Let G be a finite group and V_1, V_2 be vector spaces (of possibly different dimension), and $\rho_1 : G \to \mathrm{GL}(V_1), \rho_2 : G \to \mathrm{GL}(V_2)$ be two representations of G. Then we can define the **direct sum** of ρ_1 and ρ_2:

Definition 4 (direct sum of representations)**.** *In the above case, we define*

$$\rho : G \to \mathrm{GL}(V_1 \oplus V_2)$$

by $\rho(g) = \rho_1(g) \oplus \rho_2(g)$.

Recall that $\rho_1(g)$ and $\rho_2(g)$ are linear operators, and the direct sum of two linear operators is given by $(T \oplus U)(v, w) = (T(v), U(w))$. Let's look at an example.

Let $G = C_2$ and $V_1 = V_2 = \mathbb{C}^2$. We set ρ_1 to be the trivial representation (ie. $\rho_1(g) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for any $g \in G$), and $\rho_2(g)$ is given by

$$\rho_2(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \rho_2(1) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise 2. *Verify that these are both representations of G.*

Then, we have a representation $\rho = \rho_1 \oplus \rho_2$ given by

$$\rho(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As of right now, it is not clear that the direct sum of representations is in fact a representation. We will prove this.

Proof. Let $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ be representations of G. Let $\rho = \rho_1 \oplus \rho_2$. We will prove ρ is a representation of G.

First, $\rho(e) = \rho_1(e) \oplus \rho_2(e) = I_{V_1} \oplus I_{V_2} = I_{V_1 \oplus V_2}$.

Second, recall from linear algebra that $(A \oplus B)(C \oplus D) = AC \oplus BD$ if A and C are n-by-n matrices and B and D are m-by-m matrices. We note that $\rho(g + g') = (\rho_1(g + g') \oplus \rho_2(g + g'))$. But ρ_1, ρ_2 are representations of G, so $\rho_1(g + g') = \rho_1(g)\rho_1(g')$ and similar for ρ_2. So $\rho(g + g') = (\rho_1(g)\rho_1(g') \oplus \rho_2(g)\rho_2(g')) = (\rho_1(g) \oplus \rho_2(g)) \cdot (\rho_1(g') \oplus \rho_2(g'))$ by the fact we stated previously. But this is precisely $\rho(g)\rho(g')$, so $\rho(g + g') = \rho(g)\rho(g')$.

So overall, we see ρ is in fact a representation of G. □

The ability to combine representations is extremely important. We will see shortly that *any* representation of G can be built as a sum of so-called "irreducible" representations. This is an amazing fact in representation theory of finite groups: every group has a (finite!) collection of representations that are irreducible, and every representation is a (finite!) direct sum of these irreducible representations.

2.4 Tensor Products of Representations

If direct sums are addition for vector spaces, tensor products are multiplication for vector spaces. We know that $\mathbb{C}^n \oplus \mathbb{C}^m \cong \mathbb{C}^{n+m}$, and similarly, $\mathbb{C}^n \otimes \mathbb{C}^m \cong \mathbb{C}^{nm}$ (where $\otimes$ is the "tensor product"). Given a finite group G and two vector spaces V_1, V_2 with representations $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$, we have the "tensor product" $\rho = \rho_1 \otimes \rho_2 : G \to \mathrm{GL}(V_1 \otimes V_2)$. We will define the tensor product as follows: $\mathbb{C}^n \otimes C^m$ is defined to be $\mathbb{C}^{nm}$, and for A a n-by-n matrix and B a m-by-m matrix, we define

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \ddots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B \end{bmatrix}$$

as a block matrix. For example,

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1B & 2B \\ 0B & 1B \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Exercise 3. *Show that* $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$*, where* Tr *represents the* ***trace*** *of a matrix (the sum of the diagonal elements).*

Exercise 4. *Show that* $I_n \otimes I_m = I_{nm}$*, where* I_i *represents the i-by-i identity matrix.*

We will state without proof that, just as in the direct sum case, $(A \otimes B) \cdot (C \otimes D) = AC \otimes BD$. With this fact, we see that $\rho = \rho_1 \otimes \rho_2$ is a representation of G on $V_1 \otimes V_2$, by repreating the proof in the previous section.

2.5 Irreducible Representations

Definition 5 (G-invariant subspace)**.** *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G, and $W \subset V$ a subspace. We say W is G-**invariant** if for every $g \in G$ and $w \in W$, $(\rho(g))(w) \in W$.*

For any representation of G on a vector space V, the trivial subspace $\{0\} \subset V$ is a G-invariant subspace.

Definition 6 (irreducible representation)**.** *Let V be a representation of G. We say V is **irreducible** if V has no non-trivial G-invariant subspace.*

Let's look at some examples. Let $G = C_2$ and $\rho : G \to \mathrm{GL}(\mathbb{C}^2)$ be given by $\rho(0) = I$ and $\rho(1) = -I$. Is this irreducible? Let's look. Consider the subspace $W \subset V$ given by $W = \mathrm{span}(\{(1,0)\})$. We claim W is a G-invariant subspace of V. Let's prove this. To prove this, we need to show that for any $w \in W$ and $g \in G$, we have $(\rho(g))(w) \in W$. So pick some $w \in W$. By the construction of W, we can write $w = (w_1, 0)$. Then pick $g \in G$. If $g = 0$, we have $\rho(g) = I$ so $(\rho(g))(w) = w \in W$. Similarly, if $g = 1$, we have $\rho(g) = -I$, so $(\rho(g))(w) = -w \in W$. Thus, we have shown that for any $w \in W$ and any $g \in G$, we have $(\rho(g))(w) \in W$. Thus, W is a nontrivial G-invariant subspace of V, so V is *not* irreducible.

Definition 7 (decomposable representation)**.** *We say ρ a representation of G is **decomposable** if $\rho = \rho_1 \oplus \rho_2$ for ρ_1, ρ_2 representations of G.*

If there are no such ρ_1, ρ_2, we say ρ is **indecomposable**.

Theorem 1. *Let G be a finite group and $\rho : G \to \mathrm{GL}(V)$ a representation of G. If W is a nontrivial G-invariant subspace of V, then we must have a complement $U \subset V$ so that U is also G-invariant and $W \oplus U = V$.*

For notation, we will write $g \cdot x$ or gx to represent $(\rho(g))(x)$.

Proof. Let $p_0 : V \to W$ be the projection. Define $p(x) : V \to V = \frac{1}{|G|} \sum_{g \in G} g \cdot p_0(g^{-1}x)$. Since p_0 is the identity on W, for any $w \in W$, $g \cdot p_0(g^{-1}w) = g(g^{-1}w) = w$, since W is G-invariant so $g^{-1}w \in W$ and $p_0(g^{-1}w) = g^{-1}w$. Thus, $p(w) = w$ for any $w \in W$. Thus, $W \cap \ker p = \{0\}$. Further, for $v \in V$, $p(p(v)) = p(v)$, so $v - p(v) \in \ker(p)$. Thus, we see that $V = W \oplus \ker p$. So if we can show that $\ker p$ is a G-invariant subspace of V, we will be done.

So now, let us show $\ker p$ is G-invariant. Pick $h \in G$, $v \in \ker p$. Then

$$\begin{aligned}
p(hv) &= \frac{1}{|G|}\sum_{g\in G} g\cdot p_0(g^{-1}(hv)) \\
&= h\cdot h^{-1}\cdot \frac{1}{|G|}\sum_{g\in G} g\cdot p_0(g^{-1}\cdot (hv)) \\
&= h\cdot \frac{1}{|G|}\sum_{g\in G}(h^{-1}\cdot g)\cdot p_0((h^{-1}\cdot g)^{-1}v) \\
&= h\cdot \frac{1}{|G|}\sum_{g\in G} g\cdot p_0(g^{-1}v) \\
&= hp(v) \\
&= 0
\end{aligned}$$

Thus, $\ker p$ is G-invariant and we are done. □

The technique of defining p as "averaging over the group" is a very important and powerful tool in representation theory to produce so-called "equivariant" maps out of non-equivariant maps (a map is called **G-equivariant** if $f(g\cdot x) = g\cdot f(x)$; p_0 is not G-equivariant but p is). Note that this only works because $|G|$ is finite – this theorem is not true if G is not a finite group. This theorem also admits two powerful corollaries:

Corollary 1. *ρ is indecomposable if and only if V is irreducible.*

Corollary 2 (Maschke's Theorem)**.** *Every representation of G is a sum of irreducible representations.*

With this new technology, let us go back to our example. Let $U = \operatorname{span}(\{(0,1)\})$. By similar logic to before, U is also a G-invariant subspace of V. Further, $W \oplus U = V$. Let us look at the matrices $\rho(0), \rho(1)$:

$$\rho(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix}$$

$$\rho(1) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \oplus \begin{bmatrix} -1 \end{bmatrix}$$

Let us define $\rho_1 : G \to \mathrm{GL}(W)$ by $\rho(0) = 1$ and $\rho(1) = -1$, and $\rho_2 : G \to \mathrm{GL}(U)$ by $\rho(0) = 1$ and $\rho(1) = -1$. By the previous computation, we see that $\rho = \rho_1 \oplus \rho_2$ as a representation on $W \oplus U = V$.

Exercise 5. *Prove that every 1-dimensional representation of a group G is irreducible.*

Thus, W and U are irreducible representations of G, and we've shown that V breaks apart as a sum of irreducible representations. Maschke's Theorem, one of the most important in finite group representation theory, tells us that, amazingly, *every representation of every finite group* breaks down as a sum of irreducible representations.

2.6 Characters

We now move to *character theory.* So far, we've seen powerful tools for decomposing representations into simpler building blocks. Now, we apply this theory via *characters* in order to develop several powerful theorems on group structure.

Definition 8 (character)**.** *Let G be a finite group and $\rho : G \to \mathrm{GL}(V)$ be a representation. We define the* ***character*** *of ρ to be $\chi : G \to \mathbb{C}$ given by $\chi(g) = \mathrm{Tr}(\rho(g))$, where* Tr *represents the* ***trace*** *of a matrix (the sum of the diagonal elements).*

Exercise 6. *Show that $\chi_\rho(e) = \dim(V)$ where $\rho : G \to \mathrm{GL}(V)$.*

Proposition 1. *For any $g, h \in G$, $\chi(ghg^{-1}) = \chi(h)$.*

Proof. This is direct from the linear algebra fact that $\mathrm{Tr}(ABA^{-1}) = \mathrm{Tr}(B)$ for any two n-by-n matrices A, B (set $A = \rho(g), B = \rho(h)$). □

A function $f : G \to \mathbb{C}$ such that $f(ghg^{-1}) = f(h)$ is called a **class function**. The above proposition can be equivalently stated as "all characters are class functions." We write χ_ρ to denote the character associated to the representation ρ.

Proposition 2. *Let ρ_1, ρ_2 be representations of G.*

1. $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$
2. $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \chi_{\rho_2}$

Proof. Exercise. (Hint: see Exercise 3). □

Given G a group, we can take the collection of all class functions on G, which we denote by $\mathrm{Cl}(G)$. This can be thought of as a vector space, and its dimension is precisely the number of conjugacy classes in G. To see this, note that a class function is an arbitrary function that is constant on conjugacy classes, so a basis of this space is given by the collection of functions that take the value 1 on a specified conjugacy class and 0 otherwise. Further, we have an inner product on this vector space that turns $\mathrm{Cl}(G)$ into a complex inner product space:

$$\langle f, f' \rangle := \frac{1}{|G|} \sum_{g \in G} f(g) \overline{f'(g)}$$

Theorem 2. *Let $V_1, \ldots, V_n$ be all of the irreducible representations of G. Then $\{\chi_1, \ldots, \chi_n\}$ is an orthonormal basis in $\mathrm{Cl}(G)$.*

The equations $\langle \chi_i, \chi_j \rangle = 0$ for $i \neq j$ are called the **orthogonality relations**.

Corollary 3. *The number of irreducible representations of G is the same as the number of conjugacy classes of G.*

This corollary is very powerful in allowing us to determine the structure of a group. Finally, we have one more important theorem about the structure of groups that can only be proven using characters:

Theorem 3. *Let* $V_1, \dots, V_n$ *be all of the irreducible representations of* G*. Let* $\chi_1, \dots, \chi_n$ *be the characters of these representations. Then* $(\dim(V_1))^2 + (\dim(V_2))^2 + \cdots + (\dim(V_n))^2 = |G|$.

Let V be the "regular representation": $V = \mathbb{C}^{|G|}$ with a basis given by the elements of G, and $\rho(g)(h) = gh$ for $g, h \in G$. Essentially, we have G acting on itself. For example, if $G = C_2$ (and we write the elements as $\{e, a\}$ to avoid confusion), we have $V = \mathbb{C}^2$ with basis $\{e, a\}$ and $e \cdot (c_1 e + c_2 a) = c_1 e + c_2 a$ and $a \cdot (c_1 e + c_2 a) = c_1 a + c_2 e$. This is the **regular representation** of C_2. (In the language of group algebras, which is beyond the assumed prerequisites, this is the natural action of G on $\mathbb{C}[G]$).

Proof. Let V be the standard representation of G, and χ its character. We use without proof that $\chi(e) = |G|$ and $\chi(g) = 0$ for $g \neq e$. From this fact, we note that $\langle \chi, \chi_i \rangle = \dim(V_i)$ by Exercise 6. Thus, $\chi = \sum \dim(V_i)\chi_i$, and

$$\chi(e) = \dim(V) = |G| = \sum \dim(V_i)\chi_i(e) = \sum (\dim(V_i))^2.$$

□

2.7 Finding Character Tables

With these powerful theorems, let's look at an example of completely determining all representations of a group using character theory. Let $G = Q_8$, the quaternion group. This is a group presented by $\langle i, j, k | i^2 = j^2 = k^2 = ijk \rangle$, a group of order 8. Without listing the elements of the group, we will find all of the irreducible representations of this group using character theory.

First, we note that there are five conjugacy classes in G: $(e), (ijk), (i, -i), (j, -j), (k, -k)$. So there are 5 irreducible representations (including the trivial representation), of degrees d_1, d_2, d_3, d_4, d_5. For simplicity, we denote the trivial representation to be the first representation, so $d_1 = 1$. Second, the order of the group is 8. Now, we know $d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 8$, which means $d_1 = d_2 = d_3 = d_4 = 1, d_5 = 2$. (For simplicity, we write degrees in increasing order.) So we start filling in our character table:

	(e)	(ijk)	$(i, -i)$	$(j, -j)$	$(k, -k)$
χ_1	1	1	1	1	1
χ_2	1				
χ_3	1				
χ_4	1				
χ_5	2				

(recall $\chi(e) = \dim(V)$ and that the character for the trivial representation takes the constant value 1). We note that $(ijk)^2 = e$ in this group, so $\chi(ijk)^2 = 1$. Since $ijk \neq e$, we must have $\chi(ijk) = -\chi(e)$ for nontrivial χ.

	(e)	(ijk)	$(i,-i)$	$(j,-j)$	$(k,-k)$
χ_1	1	1	1	1	1
χ_2	1	-1			
χ_3	1	-1			
χ_4	1	-1			
χ_5	2	-2			

Since $\langle \chi_\ell, \chi_\ell \rangle = 1$, we know $\chi_{2,3,4}(i) = \pm 1$, and similar for j, k. Since we have three nontrivial one-dimensional irreducible representations, and we must satisfy our orthogonality relations, we must have exactly one of $\chi_2(i), \chi_2(j), \chi_2(k)$ is 1, and the other two are -1. So we select $\chi_2(i) = 1, \chi_2(j) = \chi_2(k) = -1$, and similar for χ_3, χ_4.

	(e)	(ijk)	$(i,-i)$	$(j,-j)$	$(k,-k)$
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	-1
χ_3	1	-1	-1	1	-1
χ_4	1	-1	-1	-1	1
χ_5	2	-2			

We can verify that the orthogonality relations are satisfied. Finally, we know that $\langle \chi_5, \chi_5 \rangle = 1$. We know $\langle \chi_5, \chi_5 \rangle = 1 = \frac{1}{|G|} \sum_{g \in G} |\chi_5(g)|^2$, so $\sum_{g \in G} |\chi_5(g)|^2 = 8$. But $\chi_5(e)^2 + \chi_5(ijk)^2 = 8$, so $\chi_5(i) = \chi_5(j) = \chi_5(k) = 0$. So we found our character table:

	(e)	(ijk)	$(i,-i)$	$(j,-j)$	$(k,-k)$
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	-1
χ_3	1	-1	-1	1	-1
χ_4	1	-1	-1	-1	1
χ_5	2	-2	0	0	0

Exercise 7. *Find the character table for D_8, the dihedral group of order 8.*

Using representation theory to find information about groups can be time consuming, but it is a very powerful tool to glean information about groups without having to list all of their elements. Listing elements of a group given its presentation can be immensely difficult, and while character theory can be tedious, it is a well-defined algorithm that will always result in a correct character table. Be careful, a character table does not completely define a group! (See Exercise 7!) But the character table provides extremely useful information about a group and its structure that can only be found through representation theory.

3 Quivers

3.1 Prerequisites

1. **representation theory of finite groups**: all the material covered in Section 2
2. **linear algebra**: vector spaces, quadratic forms

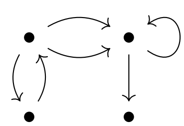
3.2 Basics of Quiver Representations

Definition 9 (quiver)**.** *A* ***quiver*** *is a graph where arrows are directed and we allow multiple arrows between vertices.*

We always require a quiver to have a finite number of vertices and a finite number of arrows.

Definition 10 (source, target)**.** *Let $\vec{Q}$ be a quiver and e an arrow in $\vec{Q}$. We say the source of e is its starting point and the* ***target*** *of e is its ending point. We write $s(e)$ and $t(e)$ respectively.*

Here is an example quiver:



We write $\vec{Q}$ to represent a quiver. Q_0 is the set of vertices in $\vec{Q}$ and Q_1 is the set of arrows in $\vec{Q}$ (quivers have arrows in them, get it?).

Exercise 8. *Write Q_0 and Q_1 for our example quiver.*

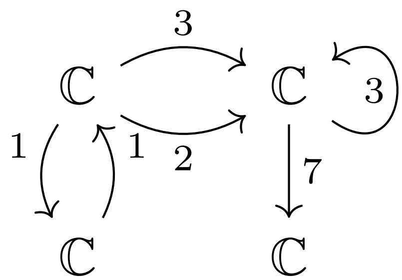
We will often want to assign an integer to every vertex in $\vec{Q}$ (we will see why very shortly), and we write $\mathbb{Z}^{Q_0}$ to represent the set of all such choices (equivalently, $\mathbb{Z}^{Q_0}$ is the collection of all functions $Q_0 \to \mathbb{Z}$). We can define a quadratic form (ie. a function that takes the place of $x \mapsto |x|^2$ in a vector space) on $\mathbb{Z}^{Q_0}$ by

$$q_{\vec{Q}}(x) := \sum_{j \in Q_0} x_j^2 - \sum_{i,j \in Q_0} d_{ij} x_i x_j$$

where d_{ij} is the number of arrows in $\vec{Q}$ that start at i and end at j. This is called the **Tits form** (after Jacques Tits, don't laugh).

Definition 11 (representation of a quiver)**.** *Let $\vec{Q}$ be a quiver. A* ***representation*** *of $\vec{Q}$ is a collection of vector spaces $\{V_i\}_{i \in Q_0}$, one for each vertex, and linear maps $\{T_e : V_{s(e)} \to V_{t(e)}\}_{e \in Q_1}$.*

Intuitively, a quiver is a template for a collection of vector spaces and linear maps between them, and a representation of a quiver is a choice of vector spaces and maps that fits into the quiver template. The following is an example of a representation of our example quiver:



Definition 12 (dimension)**.** *Given a representation* $V = \{\{V_i\}_{i\in Q_0}, \{T_e\}_{e\in Q_1}\}$ *of* $\vec{Q}$*, we say the* ***dimension*** *of* V*, written* $\dim(V)$*, is the function* $f : Q_0 \to \mathbb{Z}$ *given by* $f(i) = \dim(V_i)$*.*

For any representation V of $\vec{Q}$, $\dim(V) \in \mathbb{Z}^{Q_0}$. In our example representation, $\dim(V) = (1, 1, 1, 1)$.

Definition 13 (direct sum of representations)**.** *Let* $\vec{Q}$ *be a quiver and* V, W *be representations of* $\vec{Q}$*. We define* $V \oplus W$ *to be the representation where* $(V \oplus W)_i = V_i \oplus W_i$ *and* $T_e : V_i \oplus W_i \to V_j \oplus W_j = T_{V,e} \oplus T_{W,e}$ *(ie. the direct sum of the corresponding linear maps in* V *and* W*).*

This definition and the next one closely follow the finite group case.

Definition 14 (indecomposable representation)**.** *Let* $\vec{Q}$ *be a quiver and* V *a representation. We say* V *is* ***decomposable*** *if there are two representations* U, W *of* $\vec{Q}$ *so that* $V = U \oplus W$*, and* ***indecomposable*** *if no such* U, W *exist.*

Important: indecomposable and irreducible are *not* the same for quivers! We will only discuss indecomposable representations, but do not confuse these with irreducible representations! With some basic definitions out of the way, let's start looking at the ADE Classification of quivers.

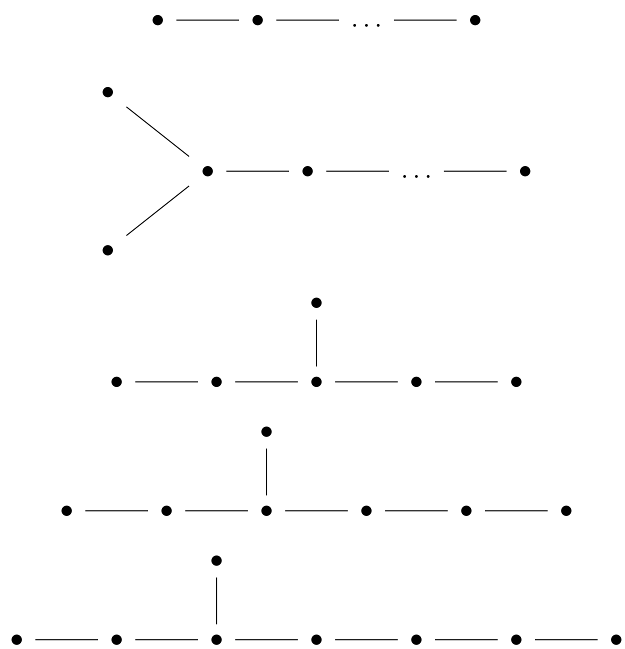
3.3 Finite-Type Quivers

Definition 15 (finite-type quiver)**.** *A quiver* $\vec{Q}$ *is said to be* ***finite-type*** *if there are only finitely many indecomposable representations of* $\vec{Q}$*, and* ***infinite-type*** *otherwise.*

Unlike the finite group case, most quivers have an infinite number of indecomposable representations. But which quivers are finite-type?

Theorem 4 (Gabriel)**.** *A quiver* $\vec{Q}$ *is finite-type if and only if its Tits form* $q_{\vec{Q}}$ *is positive definite.*

Recall that a quadratic form $q(x)$ is said to be positive definite if $q(x) > 0$ for $x > 0$. This remarkable theorem of Gabriel allows us to directly classify finite type quivers. It is easy to prove (although we will not prove it here) that the Tits form for $\vec{Q}$ is positive definite if and only if Q (the underlying graph of $\vec{Q}$, ie. the graph with no directions on the arrows) is one of the following:



The first quiver is called A_n, where n is the number of vertices. The second quiver is called D_n, where n is the number of vertices. The third quiver is E_6, the fourth is E_7, and the fifth is E_8. Together, these two infinite families and three explicit quivers are called the **ADE quivers**. They show up everywhere in math: in representation theory (here, in Lie theory, etc), in string theory, in geometry, etc. It is amazing that these quivers can be found not only by investigating the Tits form but also by considering finite-type quivers.

3.4 McKay Correspondence

We will construct a specific type of quiver called a **McKay quiver**. Let G be a finite group, $V_1, V_2, \ldots, V_n$ be the irreducible representations of G except for the trivial representation, and V any representation of G. Recall that any representation of G can be decomposed into a sum of irreducible representations. Let A_{ij} be the number of times V_i appears in the decomposition of $V \otimes V_j$. Then, we can define a quiver $\vec{Q}$ where the vertices are the indecomposable representations and there are A_{ij} arrows between any two different vertices. This is not easy for us to compute since we did not discuss the tensor product in depth, but as an example with $G = C_3$, we get the quiver A_2 as its McKay quiver.

Let's change topics drastically (the connection will appear shortly). SU(2) is the group of all 2-by-2 complex matrices that are unitary (ie. $A^*A^t = I$, where A^* is the complex conjugate of A), under multiplication. It can be thought of as the group of rotations on the surface of the sphere. A finite subgroup of SU(2) represents a finite symmetry of the sphere, ie. a regular polyhedron (Platonic solid) or a regular polygon.

We can then classify these subgroups:

1. C_n, $n \geq 2$ corresponding to the symmetries of a singular n-gon
2. D_{2n}, $n \geq 2$ corresponding to the symmetries of two n-gons attached at every edge[1]
3. T, the symmetries of the tetrahedron
4. O, the symmetries of the cube or octahedron (these are dual shapes and as such have the same symmetries)
5. I, the symmetries of the dodecahedron or icosahedron (these are dual shapes and as such have the same symmetries)

Now, we arrive at the amazing theorem:

Theorem 5 (McKay). *Finite subgroups of* SU(2) *are in bijection with ADE quivers via the construction of the McKay quiver.*

The McKay quiver of C_n is A_{n-1}, the McKay quiver of D_{2n} is D_{n+2}, the McKay quiver of T is E_6, the McKay quiver of O is E_7, and the McKay quiver of I is E_8. This is a truly extrordinary result that appears out of nowhere! There are two infinite families of ADE quivers corresponding to polygons and dihedrons, and three explicit quivers that correspond, amazingly, to the *five Platonic solids!*

[1] this is called a dihedron and is difficult to visualise, but imagine the top half of the sphere is one polygon and the bottom half of the sphere is the other polygon