Introduction to Monstrous Moonshine

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1 Introduction

Numerology is often a coincidence. It's quite rare in math that a simple equality of two numbers has a deep underlying understanding.

$$196884 = 196883 + 1$$

- John McKay, 1978

This formula looks really coincidental, but it actually connects two vastly different areas of math. The left-hand side is the 1st order coefficient of the q-expansion of the j-invariant. The right-hand side is the sum of the dimensions of the two smallest irreps of the monster group. To quote Terry Gannon (author of [Gan07], the book we are following for this seminar), "moonshine is the explanation and generalisation of this unlikely connection" [Gan04]. Now, this could just be a coincidence. We can expand the j-invariant as

$$J(\tau) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

([Gan04]) and the irreps of M have dimensions

$$1, 196883, 21296876, 842609326, \dots$$

([Slo19]).

$$\begin{aligned} 196884 &= 196883 + 1 \\ 21493760 &= 21296876 + 196883 + 1 \\ 864299970 &= 842609326 + 21296876 + 2 \cdot 196883 + 2 \cdot 1 \end{aligned}$$

So there is definitely something interesting going on here!

Today, I will give a brief overview of the important components of Moonshine and state the conjecture that we will prove this quarter.

2 What is the Monster?

In the mid to late 20th century, one of the major topics in finite group theory was finite simple groups. There was a massive effort by many leading mathematicians to fully classify the finite simple groups, and it was completed in 2004. It took nearly a hundred authors publishing tens of thousands of pages between 1955 and 2004 to finish the classification. Today, we know that the finite simple groups fit into four classes: three infinite families and 26 sporadic groups. These are

1. $\mathbb{Z}/p\mathbb{Z}$ where p is prime
2. A_n, $n \geq 5$
3. Groups of Lie type (including the Tits group)
4. One of the 26 sporadic groups

These are important for a variety of reasons, but one nice reason is that every finite group admits a composition series by simple groups, and by Jordan-Hölder, it is unique up to permutation. So classifying finite simple groups allows us to understand the composition of all finite groups. Of the sporadic groups, the largest one has order

$$|M| \approx 8 \cdot 10^{53}$$

and is called the Monster group M (a.k.a. F_1). There are a lot of ways of thinking about the Monster group (which Zach will go into detail on next week), but the original creation that I'd like to discuss is due to Griess. In 1980, Griess showed that there exists a commutative, nonassociative algebra Gr on a real vector space of dimension 196884, with $\text{Aut}(Gr) = M$. This used to produce the first full construction of M in 1982.

In particular, M fixes a vector v and acts irreducibly on $(\text{span}(\{v\}))^{\perp}$, so Gr decomposes as a sum of $\text{span}(\{v\}) \oplus Gr_{196883}$, where M acts trivially on the first factor and irreducibly on the second factor. This gives the first meaning to $196884 = 196883 + 1$: Gr decomposes as the trivial representation of M plus a simple (in fact the smallest simple) of dimension 196883.

3 What is the j-invariant?

Now, let's introduce the other key component, the j-invariant. I'll define the basics of modular curves, and explain where the J function comes from. We all know that $\text{SL}_2(\mathbb{R})$ acts on the complex upper-half plane $\mathbb{H}$ by fractional linear transformations. Really it is $\text{PSL}_2(\mathbb{R})$, but we will work with SL for convenience.

Let G be a discrete subgroup of $\text{SL}_2(\mathbb{R})$. $G \setminus \mathbb{H}$ is a complex curve. We say G is of genus g if $G \setminus \mathbb{H}$ is of genus g. The most important choice is $G = \text{SL}_2(\mathbb{Z})$, and this is of genus 0.

Definition 1 (Moonshine-type modular group)**.** *We say a subgroup* $G \subset \mathrm{SL}_2(\mathbb{R})$ *is* ***Moonshine-type*** *if it contains* $\Gamma_0(N)$ *for some* N *and* $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \in G$ *for* $t \in \mathbb{Z}$*.*

Here, Γ_0 is the level-zero congruence subgroup,

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | c \equiv 0 \mod N \right\}$$

Any Moonshine-type modular group is necessarily commensurable with $\mathrm{SL}_2(\mathbb{Z})$ (i.e. their intersection has finite index in both it and $\mathrm{SL}_2(\mathbb{Z})$).

Definition 2 (modular function)**.** *For* G *a modular group (subgroup of* $\mathrm{SL}_2(\mathbb{R})$ *commensurable with* $\mathrm{SL}_2(\mathbb{Z})$*), we can define a class of meromorphic functions* $f : \mathbb{H} \to \mathbb{C}$ *which are* ***modular*** *for* G*:* f *is stable under the action of* G *on* $\mathbb{H}$*, and for any* $A \in \mathrm{SL}_2(\mathbb{Z})$*,* $f(A \cdot z)$ *has a Fourier expansion in* $q = e^{2\pi i z}$*, given by*

$$\sum b_n q^{n/N}$$

where N, b_n *depend on* A *and at most finitely many* b_n *for* $n < 0$ *are nonzero. This is called the* ***q-expansion*** *of* f*.*

This tells us the usual facts that you may have seen as a definition of modular functions: f is meromorphic on the compact surface $G \setminus \overline{\mathbb{H}}$, where $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$, and the G-orbits of $\mathbb{Q} \cup \{\infty\}$ are called the *cusps* of G. In particular, if G is a genus-0 group of Moonshine type, then there is a unique modular function J_G, with a q-expansion of the form

$$J_G(z) = q^{-1} + \sum_{n=1}^{\infty} a_n q^n$$

and all other modular functions for G are rational functions of J_G. This J_G is called the Hauptmodul for G. In particular, for $G = \mathrm{SL}_2(\mathbb{Z})$, the Hauptmodul is

$$J_{\mathrm{SL}_2(\mathbb{Z})}(z) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3$$

Historically, for reasons we will discuss in another talk, $j(z) = J(z) + 744$ was considered instead. This number 744 will also appear again later.

4 The Conjectures

The original question of Moonshine was... "why?" Why is 196884 so close to 196883? One of the core objects in Moonshine, that we will discuss later, is the Moonshine module, an infinite-dimensional graded M-module V. It has

$$\dim(V) = \sum q^n \dim(V_n) = qJ(z)$$

where $q = e^{2\pi iz}$. This doesn't completely determine V (nor is this a construction of V), but if we for a moment assume we have such a V, we can also consider (due to Thompson) the traces

$$T_g(z) := q^{-1} \sum_{i=0}^{\infty} \chi_{V_n}(g) q^n$$

(recalling that M acts on every graded component). Taking $g = 1$ is a twist of the previous formula. The so-called "fundamental conjecture" of Conway and Norton claims that for every $g \in M$, $T_g(z)$ is the Hauptmodul of a genus-0 subgroup G_g of $\mathrm{SL}_2(\mathbb{R})$ (of Moonshine type).

The first Moonshine-esque result (due to Ogg) was that the list of primes p when $\Gamma_0(p)^+ := \left\langle \Gamma_0(p), \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \right\rangle$ had genus 0 was precisely the list of primes dividing $|M|$. More results soon came, until the conjecture of Conway-Norton was proved by Borcherds in 1992.

References

[Gan04] T. Gannon. *Monstrous Moonshine: The first twenty-five years*. 2004. arXiv: `math/0402345 [math.QA]`.

[Gan07] Terry Gannon. *Moonshine Beyond the Monster*. 2007.

[Slo19] N. J. A. Sloane. *A001379 Degrees of irreducible representations of Monster group M*. 2019.