3D Geometry

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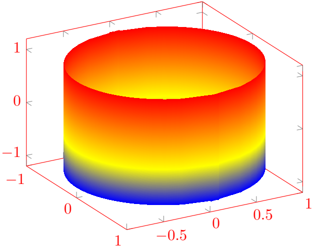
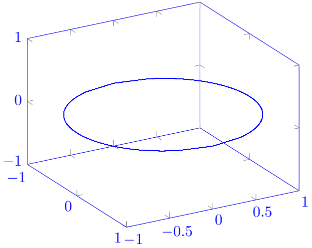
Warmup

Which of the following surfaces is the surface $x^2 + y^2 = 1$ in three dimensions?

A



B



How do you know?

Answer

The correct answer is A. Notice that I said *surface*: even though z is not written, it is still the third coordinate! To figure out what shape this will make, we notice that $x^2 + y^2 = 1$ makes a circle in the xy-plane.
Since z is not specified, it can be anything, so the resulting surface is our circle "moved up and down" along the z-axis, making a cylinder.

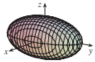
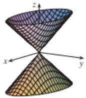
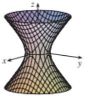
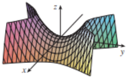
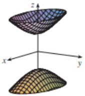
Quadric Surfaces

A **quadric surface** is a degree-two equation in x, y, and z. In general, it will look like

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

but by rotation and translation, it can be brought into a **standard form** $Ax^2 + By^2 + Cz^2 + J = 0$ or $Ax^2 + By^2 + Iz = 0$.

Types of Quadrics

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

My favourite is the hyperbolic paraboloid. I actually know how to make one out of origami :)

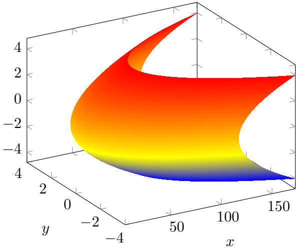
Caveat

An important thing to remember is that these are *example* equations! Consider the equation for an elliptic paraboloid for example, $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. Does the following equation represent an elliptic paraboloid?

$$\frac{x}{7} = \frac{z^2}{2} + y^2$$

Answer

Yes. We swapped x and z (which just rotates the surface without changing what it is).



We could also write it as $\frac{z}{7} = \frac{x^2}{2} + y^2$ after rotation.
https://www.desmos.com/3d/grfvgnkrqj

List of Quadrics

1. Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
2. Elliptic Paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
3. Hyperbolic Paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$
4. Cone: $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
5. Hyperboloid of One Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
6. Hyperboloid of Two Sheets: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Practice! Identify each of the following surfaces. You might need to do some algebra to put them into standard form.

1. $4x^2 - y^2 + 2z^2 + 4 = 0$
2. $x^2 + 2z^2 - 6x - y + 10 = 0$
3. $9x^2 - y^2 + z^2 = 0$

Solutions

1. $4x^2 - y^2 + 2z^2 + 4 = 0$ Hyperboloid of Two Sheets.
   Written in standard form: $-x^2 + \frac{y^2}{4} - \frac{z^2}{2} = 1$.
2. $x^2 + 2z^2 - 6x - y + 10 = 0$ Elliptic Paraboloid.
   Written in standard form: $y - 1 = (x-3)^2 + 2z^2$.
3. $9x^2 - y^2 + z^2 = 0$ Cone.
   Written in standard form: $y^2 = 9x^2 + z^2$.

Let's go over the techniques for each of these. Some might be translated or rotated, but they are still quadrics!

Coordinate Planes

What if we have a general quadric equation but we don't remember the equations for standard forms? What can we do? Let's look at an example (maybe you know what this is but bear with me):

$$x^2 + y^2 + z^2 = 1$$

Let's work this out in detail.

Coordinate Planes II

Let's look at the plane $x = 0$. This is the yz-plane, and our surface should intersect the yz-plane in a two-dimensional shape. How do we find its equation?

Since the yz-plane is $x = 0$, we can set $x = 0$ in our equation $x^2 + y^2 + z^2 = 1$ to find our shape. We will get $y^2 + z^2 = 1$, which is a circle. What do we get in the other two coordinate planes?

More circles. What shape is this?

An ellipsoid! (A sphere!)

Final Problem

21–28 Match the equation with its graph (labeled I–VIII). Give reasons for your choices.

21. $x^2 + 4y^2 + 9z^2 = 1$

22. $9x^2 + 4y^2 + z^2 = 1$

23. $x^2 - y^2 + z^2 = 1$

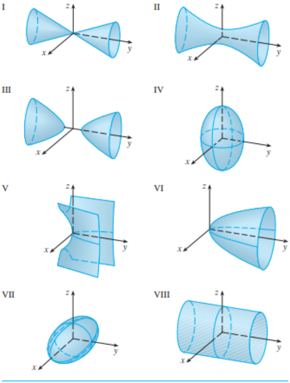
24. $-x^2 + y^2 - z^2 = 1$

25. $y = 2x^2 + z^2$

26. $y^2 = x^2 + 2z^2$

27. $x^2 + 2z^2 = 1$

28. $y = x^2 - z^2$



Answer

1. VII
2. IV
3. II
4. III
5. VI
6. I
7. VIII
8. V

More Coordinate Planes

If there's time, let's practice identifying surfaces using coordinate planes without using standard form.

$$\frac{x^2}{2} + x + y^2 + 2y + z^2 + 2z = 3$$

Answer

Let's set $x = 0$. Then we have $y^2 + 2y + z^2 + 2z = 3$. We can complete the square to get $(y+1)^2 + (z+1)^2 = 5$. This is a circle!

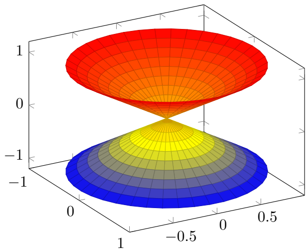
What if we set $y = 0$? We get $\frac{x^2}{2} + x + z^2 + 2z = 3$, and this is just $\frac{(x+1)^2}{2} + (z+1)^2 = 5$, an ellipse!

Guess what happens in the $z = 0$ case. (Another ellipse!)

What shape looks like an ellipse when you cut it along any axis? An ellipsoid! `https://www.desmos.com/3d/pmc0szqp7u`

Another Problem

Practice! Describe the cross-sections of a cone:



Answer

When $x = 0$, we get a hyperbola, when $y = 0$ we get a hyperbola, and when $z = 0$ we get a point.

Sketching

If we still have time, let's discuss sketching. We start by putting it in standard form, and then compute its cross-sections to get a rough sketch.

For example, take $9x^2 - y^2 + z^2 = 0$. This is in standard form, and we can see it is a cone with axis of symmetry across the y-axis. We can check the z-cross section and see it is $9x^2 = y^2$. This is a pair of lines given by $y = \pm 3x$. We can do the same for the $x = 0$ cross-section and see it is $y^2 = z^2$, so $y = \pm z$. So overall we see that we have a cone that is stretched by 3 in the z-axis, and we can sketch it approximately from this information.

Try to sketch it yourself and see what it looks like.

`https://www.desmos.com/3d/nvi6qp2q1v`

Conclusion

Thank you for coming!