Jacquet-Langlands Correspondence

Max Steinberg

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$$\begin{tikzcd}
\text{inner forms} \arrow[r, "\text{easier}"] \arrow[d, equal, "-\otimes_F \overline{F}"] & \text{reps} \arrow[d, equal, "\text{Jacquet-Langlands}"] \\
\mathrm{GL}_n \arrow[r, "\text{harder}"] & \text{reps} \arrow[d, "\text{Satake}"] \\
& \text{Hecke eigenvalues} \arrow[d, "\text{Tamagawa}"] \\
& \text{L-functions}
\end{tikzcd}$$

1 Outline

This talk will cover the Jacquet-Langlands correspondence in [JL70]. I will use this as a defining example of the Langlands philosophy, and briefly discuss how this connects to L-functions. I'd imagine it would be illegal to give a talk in this class without at least mentioning L-functions!

After motivating this result, I will follow the exposition in [Bad01] and their general outline. In particular, I will *not* follow Jacquet-Langlands's proof for GL(2). Instead, I will consider GL(2) as a starting point, but sketch a proof of the Jacquet-Langlands correspondence for GL(n) following [Bad19]'s exposition on the proof of Deligne-Kazhdan-Vignéras [DKV84].

2 Definitions

We fix F a local non-Archimedean field for this talk.

Definition 1 (semisimple element)**.** *An element $g \in G$ is called **semisimple** if its characteristic polynomial has distinct roots over $\overline{F}$. The conjugation of a semsimple element is semisimple, so we may refer to **semisimple conjugacy classes**.*

Definition 2 (inner form)**.** *Let G' be an algebraic group over F. If $G' \otimes_F \overline{F} \cong \mathrm{GL}_n(F) \otimes_F \overline{F} = \mathrm{GL}_n(\overline{F})$, then G' is called an **inner form** of $\mathrm{GL}_n(F)$.*

In particular, let r, d be two positive integers so that $rd = n$. If D is a central division algebra over F of dimension d^2, then $G' = \mathrm{GL}_r(D)$ is an inner form of $\mathrm{GL}_n(F)$. Classical algebra tells us that all inner forms of $\mathrm{GL}_n(F)$ are of this form.

Definition 3 (admissible representation)**.** *Let G be a locally compact group. If π is a representation of G on a Hilbert space H, we say π is **admissible** if V^K is finite-dimensional for any compact open subgroup $K \leq G$.*

Definition 4 (square-integrable representation)**.** *Let G be a locally compact group and H a Hilbert space. A unitary, irreducible representation π of G on H is said to be **square-integrable** if for all nonzero $\phi, \psi \in H$ we have*

$$\int_G |\langle \pi(g)\phi, \psi \rangle|^2 \, dg < \infty$$

where dg is the (right) Haar measure.

We write $E^2(G)$ to denote the set of all admissible, square integrable representations of G. These are *not* the same as automorphic representations, but they are quite related (as we will see later).

Theorem 1 (Jacquet-Langlands, [Bad01] 2.4)**.** *Let G' be an inner form of G. Then there is a correspondence*

$$E^2(G') \xrightarrow{\sim} E^2(G)$$

that is in some sense "character-preserving": if π', a representation of G', maps to π, then we must have

$$\chi_\pi(g) = (-1)^{n-r} \chi_{\pi'}(g')$$

whenever $g \leftrightarrow g'$.

This is the local formulation. This was proven in the GL(2) case by Jacquet-Langlands, GL(3) for $\mathrm{char}(F) = 0$ by Flath, $r = 1, n > 1$ by Rogawski, and finally $\mathrm{char}(f) = 0$ by Deligne-Kazhdan-Vignéras.

3 Langlands Philosophy

Why do we care about this? What is the use of this? The answer is that, on average, inner forms of $\mathrm{GL}(n)$ are vastly easier to work with and we would love more information about automorphic representations, Galois representations, L-functions, and modular forms associated to $\mathrm{GL}(n)$, as they all contain important arithmetic data. Directly finding automorphic representations of $\mathrm{GL}(n)$ would be amazing, because they are associated to a multitude of sources of valuable arithmetic data. But it can be hard to actually find these representations! Reducing the problem to $\mathrm{GL}_r(D)$ is easier most of the time.

As an example, consider the original Jacuqet-Langlands correspondence for GL(2). It relates automorphic representations of GL(2), a very valuable thing to know, to quaternion algebras. Quaternion algberas are much easier because they can be studied via Shimura curves (according to Jacob), which is something that can actually be done in practice. A lot is known about curves, not so much about automorphic representations!

This result, Jacquet-Langlands and its subsequent generalisations, forms the basis of the Langlands Programme.

4 L-Functions

I will explain some of the results on L-functions and how this result is useful, following the result of Tamagawa [Tam63], rather than Jacquet-Godement [JG72].

Definition 5 (Hecke algebra)**.** *For G a reductive group over F, the* ***Hecke algebra of*** *G, denoted $H(G)$, is given by*

$$H(G) := \{f : G \to \mathbb{C} | f \text{ is locally constant with compact support}\}$$

and is an algebra under the convolution product.

Definition 6 (Langlands dual)**.** *For a reductive group G with root datum $(X^\bullet, X_\bullet)$, the* ***Langlands dual*** *${}^LG$ is the reductive group with root datum $(X_\bullet, X^\bullet)$ (permuting the roots and coroots).*

A core tools that allows us to understand L-functions is the Satake isomorphism, that allows us to transfer eigenvalues of the Hecke algebra to local L-functions. We will work over $\mathbb{C}$ for this section. The Satake isomorphism tells us that

$$H(G) \otimes \mathbb{C} \cong R({}^LG) \otimes \mathbb{C}$$

So to any complex character of the Hecke algebra, we can associate a character of $R({}^LG)$: a character is a map $\omega : R({}^LG) \otimes \mathbb{C} \to \mathbb{C}$. But these characters are indexed by semisimple conjugacy classes in ${}^LG(\mathbb{C})$ ([Gro10] 6), so we have a correspondence between complex characters of the Hecke algebra of G and semisimple conjugacy classes in its dual. To a character ω of $H(G)$, we write $s(\omega)$ for this conjugacy class, called its *Satake parametre*.

Theorem 2 ([Gro10] 6.4)**.** *The map $\pi \mapsto s(\pi)$ gives a bijection between the set of isomorphism classes of unramified ($\dim \pi^K = 1$) irreps of G and the set of semisimple conjugacy classes in ${}^LG(\mathbb{C})$.*

If $\pi = \pi(s)$ is an unramified representation of G, and V is a complex, finite-dimensional representation of ${}^LG(\mathbb{C})$, we can define the local L-function $L(\pi, V, X)$ in $\mathbb{C}[[X]]$ by

$$L(\pi, V, X) = \det_V (1 - sX)^{-1}$$

In particular, let $G = \mathrm{GL}_n$ so that ${}^L G = \mathrm{GL}_n(\mathbb{C})$. Take $V = \mathbb{C}^n$ the standard representation. We will leave α_i undefined (they are eigenvalues of certain coroot elements [char$(K\lambda(\pi)K)$, via the Cartan decomposition for coroots] acting on π^K). Then, by the formula of Tamagawa [Tam63],

$$L(\pi, V, X) = \left(\sum_{k=0}^{n} (-1)^k q^{-k(n-k)/2} \cdot \alpha_k \cdot X^k \right)^{-1}$$

So we can get interesting calculations for local L-functions derived from our representations of G. But in general, finding representations of G can be hard, so we would like an easier way to do that!

5 Interlude

Definition 7 (orbital integral)**.** *For $g \in G$ (not actually all, only the regular semisimple, but we ignore this), the centraliser of g is a maximal torus T_g. For $f \in H(G)$, define the* ***orbital integral of f with respect to*** *g as*

$$\Phi(f; g) := \int_{G/T_g} f(xgx^{-1})\, dx$$

I will slightly switch paradigms here, from local to global. For the previous half of the talk, I only mentioned representations of G and G', and in particular never mentioned automorphic representations or adeles. These are purely local phenomena, but in order to prove this correspondence, we need to switch to the global setting for a moment.

Let K be a global field and G a reductive group over K with centre Z. For every place ν of K, let K_ν be the completion of K at ν, $G_\nu := G(K_\nu)$, and $Z_\nu = Z(K_\nu)$. Let $\mathbb{A}$ be the adeles of K. Let $\mathcal{O}_\nu$ be the ring of integers of K_ν.

We define $G(\mathbb{A})$ to be the restricted product of $G(K_\nu)$ with respect to $G(\mathcal{O}_\nu)$. Let $H(G(\mathbb{A}))$ be the Hecke algebra over $G(\mathbb{A})$. For each finite ν, we fix a Haar measure on G_ν that gives $G(\mathcal{O}_\nu)$ measure 1. For all other ν, fix an arbitrary Haar measure on G_ν. For every place ν we have $Z_\nu \cong K_\nu^*$, so fix a Haar measure that gives $\mathcal{O}_\nu^*$ measure 1. Taking the product measures, we get measues on $G(\mathbb{A})$ and $Z(\mathbb{A})$, which we will fix implicity for the remainder of this talk.

For ω a unitary character of $Z(\mathbb{A})$ trivial on $Z(K)$, we can define $L^2(G(\mathbb{A}), \omega)$ as the space of functions $\phi : G(\mathbb{A}) \to \mathbb{C}$ such that

1. ϕ is left invariant under $G(K)$
2. for $z \in Z(\mathbb{A})$ and $g \in G(\mathbb{A})$, $\phi(zg) = \omega(z)\phi(g)$
3. $$\int_{G(K)Z(\mathbb{A})\backslash G(\mathbb{A})} |\phi(g)|^2 < \infty$$

This is a Hilbert space.

Definition 8 (cuspidal form)**.** *For $\phi \in L^2(G(\mathbb{A}), \omega)$, we say ϕ is cuspidal if for every proper parabolic subgroup P of G, with N its unipotent radical, we have*

$$\int_{N(K)\backslash N(\mathbb{A})} \phi(ng)\, dn = 0.$$

We denote $L^2_c(G(\mathbb{A}), \omega)$ the subspace of cuspidal forms. $G(\mathbb{A})$ acts on $L^2(G(\mathbb{A}), \omega)$ by translations, and it fixes $L^2_c(G(\mathbb{A}), \omega)$. Let ρ_c be the representation of $G(\mathbb{A})$ on $L^2_c(G(\mathbb{A}), \omega)$ induced by the action on $L^2(G(\mathbb{A}), \omega)$. Then ρ_c is a unitary representation with central character ω. It is **not** irreducible. In fact, it decomposes discretely and every irrep summand has finite multiplicity. An irreduvible subrepresentation of ρ_c is called a *automorphic cuspidal* representation. These are the main object that we will need to fully formulate and prove the Jacquet-Langlands correspondence.

6 A Sketch of a Proof

Following [Bad19], we introduce a six-step plan to prove this theorem.

6.1 Step 1: Statement and Transfer Theory

This is what I did at the beginning of this talk, but I will elaborate now. Fix a place ν, and as in the original statement let G, G'.

1. Define an injection from conjugacy classes of G'_ν to the classes of G_ν
2. Write $g \leftrightarrow g'$ for $g \in G_\nu, g' \in G'_\nu$ if they are conjugate under the previous map
3. Write $f \leftrightarrow f'$ for $f \in H(G_\nu), f' \in H(G'_\nu)$ if $\Phi(f; g) = \Phi(f'; g')$ when $g \leftrightarrow g'$, and $\Phi(f; -)$ is zero on classes not in the image of the previous map

This allows us to *state* the local correspondence:

Theorem 3 (local Jacquet-Langlangs, [Bad19] 6.1)**.** *Let G, G' as before. Let π be an admissible, square integrable representation of G_ν. Then there is a unique up to isomorphism representation π' of G'_ν such that $\operatorname{tr} \pi(f) = \pm \operatorname{tr} \pi'(f')$ whenever $f \leftrightarrow f'$, and the sign depends only on G' (not even on the place).*

6.2 Step 2: The Trace Formula

Theorem 4 (Selberg trace formula)**.** *Let $f \in H(G(\mathbb{A}))$, such that there is a cuspidal representation π and place ν_1 so that $\operatorname{tr}(\pi(f_{\nu_1})) = 1$ and $\operatorname{tr}(\pi'(f_{\nu_1})) = 0$ for all other smooth cuspidal irreps of G. There is another required condition*

(ν_2 with f_{ν_2} supported in the elliptic set of G_{ν_2}) but we omit it for simplicity. Then, $\rho_c(f)$ has a trace, and

$$\operatorname{tr}(\rho_c(f)) = \sum_{O \in X} \mu(G_{\gamma_O}(F)Z(\mathbb{A}) \setminus G_{\gamma_O}(\mathbb{A})) \int_{Z(\mathbb{A})} \omega(z)\Phi(f, z\gamma_O)\, dz$$

where X is the set of conjugacy classes in in $Z(K) \setminus G(K)$. For $O \in X$, pick $\gamma_O \in O$ and let G_{γ_O} denote the centraliser.

The next step is to verify the previous theorem for G, G'.

6.3 Step 3: Comparing the Trace Formulae

This is the most arcane step so we will omit it for time. Put simply, we pick a nice set of $f \leftrightarrow f'$ and verify the equality geometric sides of the trace formulae manually.

6.4 Step 4: Separate Discrete Series

Let S be the set of places where G' does not split. We can write $G(\mathbb{A}) = G_S \times G_{V \setminus S}$, where

$$G_S = \prod_{\nu \in S} G_\nu, G_{V \setminus S} = \prod_{\nu \notin S} G_\nu$$

and similarly $G'(\mathbb{A}) = G'_S \times G'_{V \setminus S}$. We can write $\rho_c = \bigoplus \pi_i^{m_i}$ with π_i irreducible, so our previous step tells us

$$\sum_{i \in I} m_I \operatorname{tr} \pi_i(f) = \sum_{j \in J} m'_j \operatorname{tr} \pi'_j(f')$$

when $f \leftrightarrow f'$. Using our decomposition, we write $\pi_i = \pi_{i,S} \otimes \pi_{i,V \setminus S}$. We note that $G'_{V \setminus S} = G_{V \setminus S}$, so we can simply write I_t, J_t as the decompositions for G_S, G'_S, and we have

$$\sum_{i \in I_t} m_i \operatorname{tr} \pi_i(f) = \sum_{j \in J_t} m'_j \operatorname{tr} \pi'_j(f')$$

6.5 Step 5: Prove the Global Correspondence

Then, we prove the Strong Multiplicty-One Theorem: all $m_i, m'_j = 1$, and $|I_t| \leq 1, |J_t| \leq 1$. Thus, we can fix a discrete series representation Π of $G'(\mathbb{A})$, and set $t := \Pi'_{V \setminus S}$. Then $|I_t| = 1$ and we see there is a representation Π of $G(\mathbb{A})$, which is a discrete series representation (recall discrete series means subrep of ρ, while cuspidal is subrep of ρ_c), and

$$\operatorname{tr} \Pi_S(f) = \operatorname{tr} \Pi'_S(f')$$

and we see the **Global Jacquet-Langlands Correspondence**: we have an injective map from the set of discrete series of $G'(\mathbb{A})$ to those of $G(\mathbb{A})$ that is character-preserving.

6.6 Step 6: Prove the Local Correspondence

Finally, we see that

$$\prod_{\nu \in S} \operatorname{tr} \Pi_\nu(f_\nu) = \prod_{\nu \in S} \operatorname{tr} \Pi'_\nu(f'_\nu)$$

if $f_\nu \leftrightarrow f'_\nu \in S$. By varying only f_{ν_0}, f'_{ν_0}, we see that

$$\operatorname{tr} \pi_{\nu_0}(f_{\nu_0}) = \lambda \operatorname{tr} \pi'_{\nu_0}(f'_{\nu_0})$$

It can then be shown that $\lambda = \pm 1$ depending only on G', and we are done.

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