Fargues-Fontaine Curve

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1 Review

The general approach we will be taking today is defining the Fargues-Fontaine curve from three different viewpoints and ultimately combine all of the viewpoints into one. Let's start by recalling a couple definitions and facts from previous weeks.

Definition 1. *For a perfectoid ring R, its **tilt** is $R^\flat := \lim_\phi R$.*

Definition 2. *(Definition 5 from last week). For a perfectoid Tate ring (R, R^+), an untilt is a perfectoid Tate ring $(R^\sharp, R^{\sharp+})$ together with an isomorphism $R^{\sharp\flat} \to R$ identifying $R^{\sharp+}$ and R^+. Further, recall that Untilt is a pro-étale sheaf on Perf, as is $\mathrm{Spd}(\mathbb{Q}_p) \simeq \mathrm{Untilt}_{\mathbb{Q}_p}$.*

Definition 3. *(Definition 6.1.7 in Scholze, defined in Week 3). A **perfectoid field** is a perfectoid Tate ring R which is a nonarchimedean field. Remark: a perfectoid Tate ring such that the underlying ring is a field is in fact a perfectoid field.*

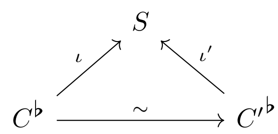
2 Classification of Untilts

2.1 Equivalences of Untilts

Recall that we previously defined the Untilt sheaf by

$$S \mapsto \left\{ (S^\sharp, \iota) | S^\sharp \text{ is a perfectoid space}, \iota : S^{\sharp\flat} \to S \right\} /\text{isomorphism}$$

Briefly, I will elaborate on isomorphism untilts. For a fixed perfectoid ring S and untilts of S: $(C, \iota_C), (D, \iota_D)$, we say $(C, \iota) \cong (C', \iota')$ if there is an isomorphism $C \cong C'$ such that the induced isomorphism $C^\flat \cong C'^\flat$ is compatible with ι, ι'. So the following picture commutes:



But a question you might have is (and the reason we are going into this), is this the correct notion of equivalent untilts?

2.2 Frobenius Equivalences

It turns out there is one other equivalence of untilts that we would like to consider. Given a fixed perfectoid ring S and an untilt of it, (C, ι), we can precompose by the Frobenius automorphism an arbitrary number of times to achieve a new untilt that is not isomorphic to (C, ι).

$$(C, \iota \circ \phi^m) \ncong (C, \iota)$$

for $m \in \mathbb{Z}$. Therefore, we say two untilts $(C, \iota), (C', \iota')$ are *Frobenius equivalent* if there is some $m \in \mathbb{Z}$ so that $(C, \iota \circ \phi^m) \cong (C', \iota')$.

3 Fargues-Fontaine Curve as the Space of Untilts

For the rest of the talk we fix some perfectoid field F that is algebraically closed, contains $\mathbb{F}_p$, and is complete with respect to a non-archimedean absolute value.

Definition 4. *Define Y_F as the set of all untilts of F. Y_F is never empty (shown last week), and let $|Y_F|$ be $Y_F/\sim$ be the set of isomorphism classes of Y_F. Then the set of Frobenius equivalence classes of $|Y_F|$ is denoted by $|Y_F|/\phi^{\mathbb{Z}}$.*

Theorem 1 (Fargues-Fontaine)**.** *There exists a complete curve X_F whose points*[1] *are in natural bijection with $|Y_F|/\phi^{\mathbb{Z}}$. Further, the point of X_F corresponding to (C, ι) has residue field C.*

3.1 Grothendieck's Theorem and Vector Bundles

The goal of a large amount of p-adic Hodge theory, local class field theory, and the local Langlands programme is to aid in the understanding of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The Fargues-Fontaine curve is an excellent example of a geometric object that provides critical understanding to the aforementioned goals.

[1] closed points

Every finite étale cover of X_F is of the form $X_F \otimes_{\mathbb{Q}_p} E$ for some finite extension E of $\mathbb{Q}_p$, so

$$\pi_1^{ét}(X_F) = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p).$$

This is an essential idea in p-adic Hodge theory and its applications to p-adic Galois representations. We unfortunately don't have time to investigate this relation deeper.

There is also a different classification of vector bundles that mirrors a theorem of Grothendieck for $\mathbb{P}^1_{\mathbb{C}}$ (in which case we would have integral λ_i).

Theorem 2 (Fargues-Fontaine)**.** *For any vector bundle E on X_F, there exists a unique sequence of (non-strictly) descending rational numbers λ_i such that $E \simeq \oplus_{i=1}^m \mathcal{O}_{X_F}(\lambda_i)$.*

4 Fargues-Fontaine Curve as a Period Ring

In the vein of analogies to $\mathbb{P}^1_{\mathbb{C}}$, let's look at period rings. We will later see some connections between the three ways to view the Fargues-Fontaine curve, but for now, let's forget everything about untilts and remember some complex analysis and algebraic geometry. We will review a couple important notions of Riemann surfaces that we will use shortly.

Working on $\mathbb{P}^1_{\mathbb{C}}$, denote ∞ as the point at infinity so that $\mathbb{P}^1_{\mathbb{C}}$ is identified with $\mathbb{C}$. Then the field of meromorphic functions on $\mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\}$ is just $\mathbb{C}(z)$ (rational functions of z). We can take the subalgebra $\mathbb{C}[z] \subset \mathbb{C}(z)$ which represents meromorphic functions regular away from ∞. For any $f \in \mathbb{C}(X)$ (the field of meromorphic functions on X), we can define its order of vanishing at any $x \in X$ as the index of the minimal nonzero term of its Laurent series centred at x, possibly infinity.

$$\mathrm{ord}_x(f) = \min\{n | a_n \neq 0\} \in \mathbb{Z} \cup \{\infty\}.$$

An important fact is that at any $x \in X$, ord_x forms a valuation on $\mathbb{C}(X)$. We have a classical degree formula

$$\sum_{x \in X} \mathrm{ord}_x(f) = 0.$$

With this formula in mind, let's try to look at the Fargues-Fontaine curve as an algebraic generalisation of the Riemann sphere.

The algebraic data of $\mathbb{P}^1_{\mathbb{C}}$ is almost entirely seen in $\mathbb{C}[z]$, except we lose the datum of ∞, which we can regain from the vanishing order function at ∞ (ie. $f \mapsto \mathrm{ord}_\infty f$), which we use as a valuation on $\mathbb{C}(z)$.

$$(\mathbb{C}[z]) + (\mathrm{ord}_\infty) \rightsquigarrow \mathbb{P}^1_{\mathbb{C}}.$$

We also adapt the degree formula from before into a new formula

$$\mathrm{ord}_\infty(f) + \sum_{\mathfrak{p}\subset\mathbb{C}[z]} \mathrm{ord}_{\mathfrak{p}}(f) = 0.$$

$\mathfrak{p}$ is a prime ideal, and all such prime ideals are generated by $z - x$ for $x \in \mathbb{C}$, which gives us ord_x as the $\mathfrak{p}$-adic valuation $\mathrm{ord}_{\mathfrak{p}}$ on $\mathbb{C}[z]$. We recall that for a polynomial f, its order of vanishing at ∞ is precisely $-\deg(f)$, so we see $-\mathrm{ord}_\infty$ forms a Euclidean function on $\mathbb{C}[z]$. Recall that a Euclidean domain $(R, \deg)$ satisfies the Euclidean division axiom:

$$\forall f \in R, g \neq 0 \in R, \exists q, r \in R | f = gq + r, \deg(r) < \deg(g).$$

We can weaken this axiom to the following (requiring all 0-degree terms to be units):

$$\forall f, g \in R, \deg(g) \geq 1, \exists q, r \in R | f = gq + r, \deg(r) \leq \deg(g).$$

Such a pair $(R, \deg)$ is called an *almost Euclidean domain*, with deg an *almost Euclidean function*. This leads us to the core definition of the algebraic geometry section of this talk:

Definition 5. *An* ***algebraic*** $\mathbb{P}^1$ *is a pair* (R, ν) *where* R *is a PID and* ν *is a valuation on* $\mathrm{Frac}(R)$ *such that* $-\nu$ *makes* $(R, -\nu)$ *an almost Euclidean domain. We say such an algebraic* $\mathbb{P}^1$ *is* ***complete*** *if it satisfies our revised degree formula,*

$$\nu(f) + \sum_{\mathfrak{p}\subset R} \mathrm{ord}_{\mathfrak{p}} f = 0$$

We've introduced a lot of theory but for good reason. In order to avoid developing four hours of period ring theory, we will state without any justification that there are $\mathbb{Q}_p$-algebras B_{crys} and B_{dR} such that

1. B_{dR} is a DVR (discrete valuation ring)
2. B_{crys} is a subring of B_{dR}
3. B_{crys} has a Frobenius endomorphism denoted as ϕ

We will later see a construction of these period rings that ties in with our diamond theory. Let $B := B_{\mathrm{crys}}^{\phi=1}$ be the ring of Frobenius-fixed points of B_{crys} and ν be the restriction of ν_{dR} to $\mathrm{Frac}(B)$.

Theorem 3 (Fargues-Fontaine)**.** *(B, ν) is an algebraic $\mathbb{P}^1$ associated to the Fargues-Fontaine curve.*

By "associated," we mean that for any curve X with an identified point $\{\infty\} \in X$ where $X \setminus \{\infty\} \simeq \mathrm{Spec}(B)$ is affine, then we say (B, ord_∞) is the associated algebraic $\mathbb{P}^1$ to a curve.

5 Fargues-Fontaine Curve as a Diamond

Let's pivot away from algebraic geometry momentarily. There is an associated adic space $\mathcal{X}_F$, called the adic Fargues-Fontaine curve, with a morphism of ringed spaces

$$\mathcal{X}_F \to X_F$$

which acts like an analytification of the scheme X_F. These schemes in fact have the same vector bundles and cohomology, but unfortunately we will not get into that. However, as we learned in previous talks, for every analytic adic space, we have an associated diamond. And ${\mathcal{X}_F}^\diamondsuit$ has a very nice structure:

Theorem 4. ${\mathcal{X}_F}^\diamondsuit \simeq \mathrm{Spd}(F)/\phi^{\mathbb{Z}} \times \mathrm{Spd}(\mathbb{Q}_p)$.

This construction of the Fargues-Fontaine curve via adic spaces generalises far better: there is a "relative Fargues-Fontaine curve" for any perfectoid space Z with characteristic p, not necessarily a perfectoid field. Now that we've seen a sneak peek of the nice structure of $\mathcal{X}_F$, let's construct it. Along the way, we will see several connections to the two previous viewpoints on the Fargues-Fontaine curve.

Definition 6. *The **infinitesimal period ring** A_{inf} is defined as $W(\mathcal{O}_F)$, the Witt ring of the ring of integers of F.*

Remark 1. *We can construct the "classical" period rings (we will skip many details since these classical rings are not the main point of this construction), B_{crys} and B_{dR}, as follows. Let $\theta : A_{\mathrm{inf}} \to \mathcal{O}_{F^\sharp}$ (relying on a choice of untilt (!)) and let ξ be the generator of the principal ideal $\ker \theta$. Then*

$$B_{\mathrm{dR}} := \mathrm{Frac}\left(\lim_n \left(A_{\mathrm{inf}}/\xi^n[1/p]\right)\right)$$

$$B_{\mathrm{crys}} := A_{\mathrm{inf}}\widehat{[\xi^n/n!]}\left[\frac{1}{p}\right]\left[\frac{1}{t}\right]$$

We say an element of $\xi = \sum [\alpha_n] p^n \in A_{\text{inf}}$ is *primitive* if $\alpha_0 \neq 0$ and some $\alpha_i \in \mathcal{O}_F^\times$. The minimal such i is called the *degree* of ξ.

Theorem 5. *There is an natural bijection between $|Y|$ and the set of ideals of A_{inf} generated by a primitive element of degree 1.*

Proof. Recall $\theta : A_{\text{inf}} \to \mathcal{O}_{F^\sharp}$ (which relies on a choice of untilt). Instead fixing an algebraically closed field K containing $\mathbb{Q}_p$ and is complete with respect to a non-archimedean absolute value, we can explicitly define θ as

$$\theta : W\left(\mathcal{O}_{K^\flat}\right) \to \mathcal{O}_K, \sum_{n\geq 0} [\alpha_n] p^n \mapsto \sum_{n\geq 0} \alpha_n{}^\sharp p^n.$$

This is in fact a ring homomorphism, although we will not verify it here (recall that checking addition rules is not easy for these constructions). Define $\mathfrak{k} := \ker\theta$. This is in fact a principal ideal, generated by $p - [p^\flat]$, $p^\flat \in K^\flat$. This element is primitive of degree 1 in $W\left(\mathcal{O}_{K^\flat}\right)$.

Now going back to our field F, select some untilt (C, ι) of F. We then have a surjection $\theta : A_{\text{inf}} = W(\mathcal{O}_F) \simeq W(\mathcal{O}_{C^\flat}) \to \mathcal{O}_C$. Then, as before, call $\mathfrak{k}_C := \ker\theta$. We claim this association is a bijection. To see this, we only need to verify $(A_{\text{inf}}/\mathfrak{k})[1/p]$ is a field that untilts F when $\mathfrak{k}$ is generated by a primitive element of degree 1. This is left as an exercise to the listener. □

Let $\mathcal{A} = \text{Spa}(A_{\text{inf}})$.

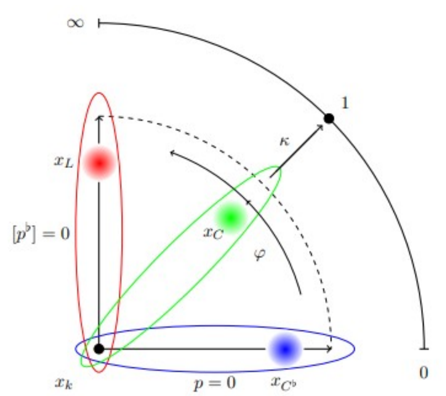


FIGURE 5. A depiction of Spa A_{inf}, where $A_{\text{inf}} = W(\mathcal{O}_{C^\flat})$. The two closed subspaces $p = 0$ and $[p^\flat] = 0$ appear as the x-axis and y-axis, respectively. We have also depicted the closed subspace $p = [p^\flat]$, which cuts out Spa $\mathcal{O}_C$, as a green ellipse. The unique non-analytic point x_k of Spa A_{inf} appears at the origin. Its complement in Spa A_{inf} is the adic space $\mathcal{Y}$, on which the continuous map $\kappa\colon \mathcal{Y} \to [0, \infty]$ is defined. The automorphism φ of Spa A_{inf} rotates points towards the y-axis, as per the equation $\kappa \circ \varphi = p\kappa$.

Then we can create $\mathcal{Y} := \mathcal{A} - (p[p^\flat])$. This is an adic space, so we naturally wonder what its diamond is. In fact,

$$\mathcal{Y}^\diamond \simeq \mathrm{Spd}(F) \times \mathrm{Spd}(\mathbb{Q}_p).$$

Proof. We will use without proof that Spd is a fully faithful functor. Via a functor-of-points approach, we will show that $\mathcal{Y}^\diamond$ and $\mathrm{Spd}(F) \times \mathrm{Spd}(\mathbb{Q}_p)$ have the same $\underline{S}$ -points for all perfect Tate-Huber pairs $\underline{S}$. By full faithfullness of Spd, an $\underline{S}$-point of $\mathrm{Spd}(F)$ is a morphism of Tate-Huber pairs $f : (F, \mathcal{O}_F) \to (S, S^+)$. As we saw last week, $\mathrm{Spd}(\mathbb{Q}_p)$ as a sheaf is equivalent to the $\mathrm{Untilt}_{\mathbb{Q}_p}$ sheaf on Perf (and all characteristic-0 fields we consider here are over $\mathbb{Q}_p$), so an $\underline{S}$-point of $\mathrm{Spd}(\mathbb{Q}_p)$ is an untilt $(\underline{T}, \iota)$ of $\underline{S}$. By construction of $\mathcal{Y}$, an $\underline{S}$-point of $\mathcal{Y}$ is merely an untilt of $\underline{S}$, $(\underline{T}, \iota)$, together with a morphism $A_{\mathrm{inf}} \to T^+$ that can be extended to $A_{\mathrm{inf}}[1/p, 1/[\pi]] \to T$. Thus, we need to construct this morphism.

$$A_{\mathrm{inf}} = W(\mathcal{O}_F) \to_f W(S^+) \simeq_\iota W\left(T^{+\flat}\right) \to_\theta T^+.$$

By the universal property of Witt vectors, every such homomorphism factors this way, so this is an isomorphism. □

This is quite close to what we want for $\mathcal{X}$. Luckily, the Frobenius from A_{inf} induces a totally discontinuous action on $\mathcal{Y}$, so $\mathcal{X} := \mathcal{Y}/\phi^{\mathbb{Z}}$ is well-defined. Then

$$\mathcal{X}^\diamond \simeq \mathrm{Spd}(F)/\phi^{\mathbb{Z}} \times \mathrm{Spd}(\mathbb{Q}_p).$$