Differential Forms

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Introduction

These notes are written by me (Max Steinberg) and my DRP students. They are not intended to be full course notes; rather, they serve as a companion to the weekly meetings of the DRP, and contain exercises that I assigned weekly. It also contains solutions written by me and my students. The goal is that eventually, through the work of me and several sets of DRP students, this document will serve as a basic outline of the theory of differential forms, with exercises and a solution manual, fit for use as a companion to a course.

What are differential forms? Differential forms are a framework to encapsulate all of the different results from calculus, in any dimension. When you took calculus, you might remember seeing a lot of similar-looking results, like Stokes's Theorem and Gauss's Theorem (the Divergence Theorem).

$$\iint_D \nabla \times \vec{F} = \int_{\partial D} \vec{F}$$

$$\iiint_D \nabla \cdot \vec{F} = \iint_{\partial D} \vec{F}$$

In the language of differential forms, these results, and many more, are encapsulated by one theorem:

Theorem 1 (Stokes-Cartan Theorem)**.**

$$\int_D d\omega = \int_{\partial D} \omega$$

Similarly, you probably learned about the dot and cross products in the first few weeks of multivariable calculus. If $v, w \in \mathbb{R}^3$ are vectors, then $v \cdot w$ is a scalar that measures how much v and w are parallel, and $v \times w$ is a vector that is perpendicular to both v and w. In the language of differential forms, $v \cdot w$ is given by $v \wedge *w$, and this can be directly interpreted as how parallel v and w are. Similarly, $v \times w$ is given by $v \wedge w$, which denotes the plane containing v and w. Recall that a plane is defined by its perpendicular vector – the plane containing v and w is equivalent to the vector $v \times w$.

Differential forms also provide a conceptual framework in which to understand concepts that might otherwise feel unmotivated or arbitrary, such as curl. I've taught and tutored for several calculus courses, including MATH 120 at Yale, and being totally honest, I still don't remember the formula for curl. But I understand it in differential forms, so I can derive it and compute it whenever I need it, and I understand how it fits into the bigger picture of three-dimensional geometry. Hopefully by the end of these notes you will too!

1 Orientation and Wedge Product

1.1 Orientation

Differential forms, among their other uses, give us a formulaic way to work with orientation. Let $v, w \in \mathbb{R}^3$ be vectors. How can we think about the orientation of the plane generated by v and w? There are two important things we want an orientation to satisfy: if we swap the order, there should be a minus sign, and if $v = w$, then the orientation is zero.

Exercise 1. *Explain in pictures why both of these requirements make sense.*

From calculus, you might remember an operation that takes two vectors, v and w, that satisfies both of these axioms: $v?w = -w?v$ and $v?v = 0$. The cross product! $v \times w = -w \times v$ and $v \times v = 0$.

Exercise 2. *Actually, if $v \times w = -w \times v$, then we already have that $v \times v = 0$. Prove this!*

In fact, the cross product satisfies another even stronger property that is quite important: it is bilinear. That is, $(c\vec{v} + \vec{w}) \times \vec{u} = c(\vec{v} \times \vec{u}) + (\vec{w} \times \vec{u})$. In fact, up to swapping arguments and scaling, the cross product is the unique bilinear map $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ that satisfies $v \times w = -w \times v$[1]. In Chapter 3, we will use some linear algebra to further this idea.

Since the cross product is bilinear, what if we tried to make a vector space out of all the possible cross products? Since the cross product is a surjective map $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, we know the answer will just be $\mathbb{R}^3$. But this provides an interesting new basis of $\mathbb{R}^3$: $\{i \times j, j \times k, k \times i\}$. We that this is just the same as $\{k, i, j\}$, but thinking about things this way allows us to extend to higher dimensions more easily.

1.2 Wedge Product

So let's define this in any dimension. Fix $n > 0$, and let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$. Fix some $0 \leq k \leq n$. Let V be the vector space with basis

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\}_{1 \leq i_1, i_2, \ldots i_k \leq n}$$

subject to $e_i \wedge e_i = 0, e_i \wedge e_j = -e_j \wedge e_i$. The symbol $\wedge$ is called the *wedge product*, and we simply use it here to combine vectors. It doesn't do anything other than say "these two symbols are next to each other" (like the hyphen in english).

Exercise 3. *Prove that V has dimension $\binom{n}{k}$. Does this make sense if $k = 0$?*

[1]Under the canonical (fixing a basis) identification $\bigwedge^2 \mathbb{R}^3 \cong \bigwedge^1 \mathbb{R}^3$ given by the Hodge star, this can be written as a map $\bigwedge^1 \mathbb{R}^3 \otimes \bigwedge^1 \mathbb{R}^3 \to \bigwedge^2 \mathbb{R}^3$ that is alternating, and thus a $\bigwedge \mathbb{R}^3$-algebra morphism. The hom-space is 1-dimensional, generated by the wedge (cross) product.

For any $v \in V$, we say that v is a *k-vector*. A 1-vector is just a vector (why?). A 0-vector is a scalar (why?).

A k-vector denotes a specific orientation for a k-dimensional object in $\mathbb{R}^n$. So for example, a plane in $\mathbb{R}^3$ is 2-dimensional, so a 2-vector represents its orientation. The unit ball in $\mathbb{R}^3$ is 3-dimensional, and a 3-vector represents its orientation.

Example 1. *Let $n = 3, k = 2$. Consider $u, v \in \mathbb{R}^3$. By the previous discussion, we can say that u and v are 1-vectors. The element $u \wedge v \in V$ is a 2-vector that corresponds to the plane containing u and v. Using the right-hand rule, we can determine which direction $u \times v$ (the cross product) sits. Writing $v \wedge u \in V$ is a different 2-vector than $u \wedge v$, which corresponds to using v first and then u in the right-hand rule.*

Exercise 4. *Pick two vectors $u, v \in \mathbb{R}^3$. Directly compute $u \times v$ and $u \wedge v$. Compare your results.*

Exercise 5. *Let ω be a k-vector and θ a ℓ-vector on $\mathbb{R}^n$, where $n > k+\ell$. Prove that*

$$\omega \wedge \theta = (-1)^{k\ell}\theta \wedge \omega$$

Recall that $\binom{n}{k} = \binom{n}{n-k}$. From our knowledge of linear algebra, this means these vector spaces are isomorphic! We can construct an explicit isomorphism:

Example 2. *Let $n = 3, k = 2$. We can construct an isomorphism $\phi : V(3, 2) \cong V(3, 1)$ given by $\phi(e_1 \wedge e_2) = e_3, \phi(e_3 \wedge e_1) = e_2, \phi(e_2 \wedge e_3) = e_1$.*

Definition 1 (Hodge star)**.** *Write $*a := \phi(a)$. This is called the **Hodge star** of a.*

Theorem 2. *Let a, b be vectors in $\mathbb{R}^n$. Then*

$$a \wedge *b = \langle a, b\rangle\, e_1 \wedge e_2 \wedge \cdots \wedge e_n.$$

Exercise 6. *Explicitly find the isomorphism from $V(4, 1) \cong V(4, 3)$, using the previous theorem.*

Exercise 7. *Think about the Hodge star on $\mathbb{R}^3$. We found the inner (dot) product – how can we also find the cross product? Hint: how did we start this discussion?*

2 Differential Forms

2.1 Covectors

We will finally define differential forms! But first, we need to introduce *covectors*.

Let V be a finite-dimensional real vector space. Recall that we can construct the *linear dual* V^* consisting of linear operators $V \to \mathbb{R}$. We call an element $v \in V^*$ a *linear functional*[2] or *covector*. Since $\dim(V) < \infty$, let $n = \dim(V) = \dim(V^*)$. We have a non-canonical (basis-dependent) isomorphism between V and V^*. Take a moment to think about what the isomorphism is. *Hint:* $\langle v, w \rangle = v^t w$.

The answer is the transpose. If v is a vector of dimension n, then it can be considered a n-by-1 matrix, so v^t is a 1-by-n matrix. A linear map $\phi : V \to \mathbb{R}$ is a map from $\mathbb{R}^n \to \mathbb{R}$, so it has size 1-by-$n$. So ϕ^t is a n-by-1 matrix, or a vector. So the map $V \mapsto V^*$ given by the transpose is invertible (given by the transpose once again), so it is an isomorphism.

So now, let's consider n-covectors. These are defined identically to n-vectors but with V^* in place of V. Once again, a 0-covector is a scalar (coscalars are just scalars, since the transpose of a 1-by-1 matrix is itself), and a 1-covector is just a covector.

Let $V = \mathbb{R}^n$ for simplicity. We have a standard basis $\{x_1, x_2, \ldots, x_n\}$, and we have a resulting *dual basis* $\{dx_1, dx_2, \ldots, dx_n\}$, where $dx_1 = x_1^t$.[3]

Exercise 8. *Verify that this is actually the dual basis. That is, show that* $dx_i(x_j) = \delta_{ij}$.

Example 3. *The space of* 2*-covectors for* V *has a basis* $\{dx_1 \wedge dx_2, dx_1 \wedge dx_3, \ldots, dx_1 \wedge dx_n, dx_2 \wedge dx_3, \ldots, dx_2 \wedge dx_n, dx_3 \wedge dx_4, \ldots, dx_{n-1} \wedge dx_n\}$ *with* $\binom{n}{2} = \frac{n(n-1)}{2}$ *elements.*

2.2 Differential Forms

Definition 2 (differential form)**.** *Let* $n > 0$ *and* $0 \le k \le n$*. A* ***k-differential form on*** $\mathbb{R}^n$*, also written as* ***k-form on*** $\mathbb{R}^n$*, or just a* ***k-form*** *when n is clear, is a* k*-covector on* $\mathbb{R}^n$*, but with coefficients in* $C^\infty(\mathbb{R}^n)$ *rather than* $\mathbb{R}$*.*

First of all, $C^\infty(\mathbb{R}^n)$ represents smooth functions on $\mathbb{R}^n$: functions which are differentiable everywhere, infinitely many times.[4] Second, what does this definition even mean?

From the previous example, we saw that $dx_1 \wedge dx_2$ was an example 2-covector on $\mathbb{R}^n$. This is also a 2-form on $\mathbb{R}^n$. So is $\sin(x_1)\, dx_1 \wedge dx_2$, which is *not* a 2-covector. So just like how space of k-covectors on $\mathbb{R}^n$ was defined as the $\mathbb{R}$-span

[2] I strongly dislike this terminology, and will not use it.

[3] I usually use e_i for the standard basis. The use of x will become clear once we introduce functions.

[4] They may not be analytic. $C^\omega(\mathbb{R}^n)$ denotes analytic functions.

of a certain basis, the space of k-forms on $\mathbb{R}^n$ is the $C^\infty(\mathbb{R}^n)$-span of the same basis.[5]

Example 4. *Let* $\omega = \sin(x)\,dx + \cos(xy)\,dy - e^{z-x}\,dz$. *This is a* 1*-form on* $\mathbb{R}^3$, *where our basis is* $\{x, y, z\}$.

We will often write our differential forms in summation notation to remain general. So we may say ω is a k-form on $\mathbb{R}^n$ given by

$$\omega = \sum_{I^k} f_I\,dx_I.$$

We often omit the k in I^k when k is clear.

Definition 3 (multi-indices)**.** *When we write this, we mean*

$$\sum_{i_1,i_2,\ldots,i_k} f_{i_1,i_2,\ldots,i_k}\,dx_{i_1} \wedge dx_{i_2} \cdots \wedge dx_{i_k}.$$

Example 5. *Using our multi-indexing notation, we can take an example* 2*-form on* $\mathbb{R}^3$, $\omega = x\,dy \wedge dz + y\,dx \wedge dz + z\,dx \wedge dy$. *Then* $\omega = \sum_I f_I\,dx_I$, *and* $f_{1,2} = z, f_{1,3} = y, f_{2,3} = z$.

Exercise 9. *Write a* 1*-form, a* 2*-form, and a* 3*-form on* $\mathbb{R}^3$. *For each, describe the functions you chose and what indices they correspond to. (For example, if* $\omega = 1\,dx$, *then* $f_1 = 1, f_2 = 0, f_3 = 0$*).*

Exercise 10. *Let* θ *be a* 1*-form on* $\mathbb{R}^3$ *and* ω *be a* 2*-form on* $\mathbb{R}^3$. *What is* $\theta \wedge \omega$*?*

[5] Since $C^\infty(\mathbb{R}^n)$ is a $\mathbb{R}$-algebra, this says that $\Omega^k(\mathbb{R}^n)$ (k-forms) is just $\bigwedge^k(\mathbb{R}^n) \otimes_{\mathbb{R}} C^\infty(\mathbb{R}^n)$.

3 * Tensor Products and the Exterior Algebra

This section is a little bit algebraic and formal, so if you would like, you can take the result of Exercise 15 and Theorem 3 as god-given and skip this section entirely.

Recall from linear algebra that given two vector spaces V and W (all vector spaces we ever consider will be finite-dimensional), we can form their *direct sum* $V \oplus W$ satisfying $\dim(V \oplus W) = \dim(V) + \dim(W)$. $V \oplus W$ has a basis $\{(v_i, 0), (0, w_j)\}$ where $\{v_i\}$ are a basis of V and $\{w_j\}$ are a basis of W. So a question might be, is there a vector space $V?W$ so that $\dim(V?W) = \dim(V)\dim(W)$?

The answer is yes: $V \otimes W$ (read "V tensor W") satisfies $\dim(V \otimes W) = \dim(V) + \dim(W)$. It is defined with a basis $\{v_i \otimes w_j\}$, and hence has ij elements in its basis. We define $v \otimes w + v' \otimes w = (v + v') \otimes w$ and similar for $v \otimes w + v \otimes w'$, but $v \otimes w + v' \otimes w'$ cannot be simplified. For scalar multiplication, $c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$.

Example 6. *Let $V = \mathbb{C}^2$ be the 2-dimensional complex vector space. We can give it a basis $\{(1, 0), (0, 1)\}$. Then $V \otimes V$ has basis $\{(1, 0) \otimes (1, 0), (1, 0) \otimes (0, 1), (0, 1) \otimes (1, 0), (0, 1) \otimes (0, 1)\}$. If we have $(a, b) = a(1, 0) + b(0, 1)$, then*

$$\begin{aligned}(a, b) \otimes (c, d) &= (a(1, 0) + b(0, 1)) \otimes (c(1, 0) + d(0, 1)) \\ &= a(1, 0) \otimes c(1, 0) + a(1, 0) \otimes d(0, 1) + b(0, 1) \otimes c(1, 0) + b(0, 1) \otimes d(0, 1) \\ &= ac((1, 0) \otimes (1, 0)) + ad((1, 0) \otimes (0, 1)) + bc((0, 1) \otimes (1, 0)) + bd((0, 1) \otimes (0, 1))\end{aligned}$$

We can write this as a matrix, $\begin{pmatrix} ac & bc \\ ad & bd \end{pmatrix}$.

Exercise 11. *Let V be the vector space of 2-by-2 real matrices. We know $\dim V = 4$, so $\dim V \otimes V = 16$. Let $v, w \in V$ with $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, w = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Write $v \otimes w \in V \otimes V$ as a 4-by-4 matrix.*

Hint: do you know any other operations on matrices that use the $\otimes$ symbol?

We can define *symmetric* and *alternating* (also known as *antisymmetric*) tensors. Let $\sigma : V \otimes V \to V \otimes V$ be given by $\sigma(v \otimes v') = v' \otimes v$ (just swapping the order). We say a tensor $\alpha \in V \otimes V$ is *symmetric* if $\sigma(\alpha) = \alpha$, and *alternating* if $\sigma(\alpha) = -\alpha$.

Define $S^2(V)$ as the vector space of symmetric tensors in $V \otimes V$, and $\bigwedge^2(V)$ as the vector space of antisymmetric tensors in $V \otimes V$.

Exercise 12. *Prove that $V \otimes V = S^2 V \oplus \bigwedge^2 V$.*

In general, we can define $S^k V$ as the space of k-symmetric tensors in $V^{\otimes k} = V \otimes \cdots \otimes V$ (k times). k-symmetric means that for any permutation $\sigma \in S_k$ (the group of permutations on k elements), $\sigma(\alpha) = \alpha$. For alternating tensors, we say $\sigma(\alpha) = (\operatorname{sgn}\sigma)\alpha$, where $\operatorname{sgn}\sigma$ denotes $(-1)^p$, with p the number of swaps σ makes.

It is still true that $V^{\otimes k} = S^k V \oplus \bigwedge^k V$.

Exercise 13. *Let V be a vector space of dimension n. Prove that $\dim \bigwedge^k(V) = 0$ if $k < 0$ or $k > n$.*

Exercise 14. *Let V be a vector space of dimension n. Prove that $\dim \bigwedge^k(V) = \dim \bigwedge^{n-k}(V) = \binom{n}{k}$.*

Exercise 15. * *Let $e_i \in \mathbb{R}^n$ denote the i-th standard basis vector. For $I = \{i_1, \dots, i_k\}$, write $e_I := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$. Let $\mathcal{I}_k = \{1 \le i_1 < i_2 < \cdots < i_k \le n\}$ (the set of all ordered sets of k elements from 1 to n: if $k = 2, n = 3$ then $\mathcal{I} = \{\{1,2\},\{1,3\},\{2,3\}\}$). Prove that $\{e_I\}_{I \in \mathcal{I}_k}$ is a basis of $\bigwedge^k(\mathbb{R}^n)$.*

It turns out that we can create a *algebra* out of $\bigwedge^k V$. Define

$$\bigwedge V := \bigoplus_{k \in \mathbb{Z}} \bigwedge^k V$$

with multiplication given by $v \times w := v \otimes w$. We will denote this by $\wedge$ (the "wedge product"), and hence write $v \wedge w$ for the multiplication.

Exercise 16. *Let $v \in \bigwedge^k V, w \in \bigwedge^\ell V$. For what value of $p \in \mathbb{Z}$ do we have $v \wedge w \in \bigwedge^p V$?*

Exercise 17. *Check that $\bigwedge V$ is well-defined as an algebra. What is its dimension in terms of $\dim V$?*

Definition 4 (smooth functions)**.** *Let $C^\infty(V)$ denote the real vector space of **smooth functions**: functions $V \to \mathbb{R}$ that are infinitely differentiable everywhere.*

Differential forms are precisely alternating tensors with coefficients in $C^\infty(V)$ rather than in $\mathbb{R}$. We can express this formally:

Definition 5 (differential form)**.** *A **differential k-form on** $\mathbb{R}^n$ ω is an element $\omega \in \Omega^k(\mathbb{R}^n) := C^\infty(\mathbb{R}^n) \otimes \bigwedge^k((\mathbb{R}^n)^*)$.*

Note that since $C^\infty(\mathbb{R}^n)$ is infinite-dimensional (as a real vector space), so is $\Omega^k(\mathbb{R}^n)$. This definition is a bit convoluted, so let's unpack it.

We defined $\Omega^k(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \otimes \bigwedge^k((\mathbb{R}^n)^*)$. This is a bit of a complicated definition, but we can work with it by unpacking it a little bit. Let $\{v_i\}_{i=1}^{\binom{n}{k}}$ be a basis of $\bigwedge^k((\mathbb{R}^n)^*)$.

Theorem 3. * *Let $\omega \in \Omega^k(\mathbb{R}^n)$. We can uniquely write*

$$\omega = \sum_{v_i} f_i(x_1, x_2, \dots, x_n)\, v_i$$

Exercise 18. * *Prove this theorem.*

By Exercise 15, we can describe (one choice of) $\{v_i\}$. Let us write dx_i to be the dual basis to e_i, in the sense that $\{dx_i\}$ is a basis of $(\mathbb{R}^n)^*$ where $dx_i(e_j) = \delta_{ij}$. Then we can write *

$$\omega = \sum_{I \in \mathcal{I}_k} f_I(x_1, \ldots, x_n)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}.$$

This is the formal construction of differential forms that satisfies all of the properties we want it to.

Exercise 19. * *Verify that this formal algebraic object actually contains differential forms.*

4 Exterior Derivative

So far we've talked a lot about "forms" and not a lot about differentials. Let's fix that. Given a differential form ω, we would like to define the derivative of ω, which we write as $d\omega$. Based on how we define a k-form, it makes sense to say that if if ω is a k-form, then $d\omega$ is a $k+1$-form. So let's think about how we can define d.

What's the first thing we know about derivatives? That the derivative of a constant is 0. So if we write $\omega = \sum f_I \, dx_I$ like usual, and all f_I are constant functions, we can say that $d\omega = 0$.

Second, we know that the derivative is linear. This means that if ω and θ are both k-forms on $\mathbb{R}^n$, and $a, b \in \mathbb{R}$, then $d(a\omega + b\theta) = a \cdot d\omega + b \cdot d\theta$. Third, we know that the derivative of a scalar-valued function is its gradient. In the language of forms, if $\omega = f$ is a 0-form, then $d\omega$ should correspond to ∇f. Recall that we can make a map between vectors and 1-forms by replacing x with dx, y with dy, etc. So if $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar-valued function, then $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is given by $(f_{x_1}, f_{x_2}, \dots, f_{x_n})$, so $d\omega = f_{x_1} \, dx_1 + f_{x_2} \, dx_2 + \cdots + f_{x_n} \, dx_n$.

Exercise 20. *Explicitly calculate $d\omega$, where ω is the 0-form on $\mathbb{R}^4$ given by $\omega = \sin(x) + y + \cos(z) + e^w$.*

With all of these rules in place, we are ready to define d! [6] Let $\omega = \sum f_I \, dx_I$ be a k-form on $\mathbb{R}^n$. We can then calculate $d\omega$:

$$\begin{aligned}
d\omega &= d\left(\sum f_I \, dx_I\right) && \text{(Definition of } \omega.) \\
&= \sum d(f_I \, dx_I) && \text{(Linearity of } d.) \\
&= \sum (df_I) \, dx_I && \text{(Because } dx_I \text{ is a constant, so we can pull it out.)} \\
&= \sum \left(\sum_{j=1}^{n} \frac{\partial f_I}{\partial x_j} \, dx_j \right) dx_I && \text{(Derivative of a 0-form.)}
\end{aligned}$$

Let's look at an example.

Example 7. *Let $\omega = (z + \cos(y)) \, dx \wedge dy + (x + e^z) \, dy \wedge dz + (y + \cosh(xz)) \, dz \wedge dx$. Then we can compute $d\omega$:*

[6] Notice that we never mentioned the product rule at all. We will come back to this in an exercise – the product rule needs a slight modification.

$$
\begin{aligned}
d\omega &= d((z+\cos(y))\,dx\wedge dy) + d((x+e^z)\,dy\wedge dz) + d((y+\cosh(xz))\,dz\wedge dx) \\
&= \left(\frac{\partial(z+\cos(y))}{\partial x}\,dx + \frac{\partial(z+\cos(y))}{\partial y}\,dy + \frac{\partial(z+\cos(y))}{\partial z}\,dz\right) dx\wedge dy \\
&+ \left(\frac{\partial(x+e^z)}{\partial x}\,dx + \frac{\partial(x+e^z)}{\partial y}\,dy + \frac{\partial(x+e^z)}{\partial z}\,dz\right) dy\wedge dz \\
&+ \left(\frac{\partial(y+\cosh(xz))}{\partial x}\,dx + \frac{\partial(y+\cosh(xz))}{\partial y}\,dy + \frac{\partial(y+\cosh(xz))}{\partial z}\,dz\right) dz\wedge dx \\
&= (0\,dx - \sin(y)\,dy + 1\,dz)\,dx\wedge dy \\
&+ (1\,dx + 0\,dy + e^z\,dz)\,dy\wedge dz \\
&+ (z\sinh(xz)\,dx + 1\,dy + x\sinh(xz)\,dz) dz\wedge dx \\
&= 1\,dz\wedge dx\wedge dy && \text{(Because } dx\wedge dx = dy\wedge dy = 0.\text{)} \\
&+ 1\,dx\wedge dy\wedge dz && \text{(Because } dy\wedge dy = dz\wedge dz = 0.\text{)} \\
&+ 1\,dy\wedge dz\wedge dx && \text{(Because } dx\wedge dx = dz\wedge dz = 0.\text{)} \\
&= 1\,dx\wedge dy\wedge dz + 1\,dx\wedge dy\wedge dz + 1\,dx\wedge dy\wedge dz \\
& && \text{(Since we make an even number of swaps each time.)} \\
&= 3\,dx\wedge dy\wedge dz
\end{aligned}
$$

Exercise 21. *Here, ω is a 2-form. Remember the Hodge star operator we talked about last time: $*\omega = (x+e^z)\,dx + (y+\cosh(xz))\,dy + (z+\cos(y))\,dz$. If we treat this as a vector field, we can write $*\omega$ as $f(x,y,z) = (x+e^z, y+\cosh(xz), z+\cos(y))$. Can you relate $d\omega$ to f somehow?*

Exercise 22. *Let $\theta = 1\,dx + \sin(xz)\,dy + y\,dz + w\,dw$, a 1-form on $\mathbb{R}^4$. Find $d\theta$.*

Now, let's work out what all the derivatives are on $\mathbb{R}^3$. We recall that $\dim \Omega^k(\mathbb{R}^3) = \binom{3}{k}$, so 0-forms are 1-dimensional, 1-forms are 3-dimensional, 2-forms are 3-dimensional, and 3-forms are 1-dimensional. We already know a few differential operators on $\mathbb{R}^3$ – what number of dimensions do they operate on? The gradient takes a scalar function and outputs a vector field, so we can say it goes $1 \to 3$. The curl takes a vector field and produces a vector field, so it goes $3 \to 3$. And finally, the divergence goes $3 \to 1$. So we might guess that the gradient is $d : \Omega^0 \to \Omega^1$, the curl is $d : \Omega^1 \to \Omega^2$, and the divergence is $d : \Omega^2 \to \Omega^3$. Let's prove this.

We already know that if $\omega = f$ is a 0-form, then $d\omega$ corresponds to ∇f, so the gradient is done. Now, let $\omega = f_1\,dx + f_2\,dy + f_3\,dz$. Then

$$
\begin{aligned}
d\omega &= (f_{1x}\,dx + f_{1y}\,dy + f_{1z}\,dz)\,dx \\
&\quad + (f_{2x}\,dx + f_{2y}\,dy + f_{2z}\,dz)\,dy \\
&\quad + (f_{3x}\,dx + f_{3y}\,dy + f_{3z}\,dz)\,dz \\
&= -f_{1y}\,dx \wedge dy + f_{1z}\,dz \wedge dx + f_{2x}\,dx \wedge dy - f_{2z}\,dy \wedge dz - f_{3x}\,dz \wedge dx + f_{3y}\,dy \wedge dz \\
&= (f_{2x} - f_{1y})\,dx \wedge dy + (f_{3y} - f_{3z})\,dy \wedge dz + (f_{1z} - f_{3x})\,dz \wedge dx
\end{aligned}
$$

Which is just the Hodge star of $\nabla \times (f_1, f_2, f_3)$. So d on 1-forms is just the curl! Now, let $\omega = f_1\,dy \wedge dz + f_2\,dz \wedge dx + f_3\,dx \wedge dy$. Then

$$
\begin{aligned}
d\omega &= (f_{1x}\,dx + f_{1y}\,dy + f_{1z}\,dz)\,dy \wedge dz \\
&\quad + (f_{2x}\,dx + f_{2y}\,dy + f_{2z}\,dz)\,dz \wedge dx \\
&\quad + (f_{3x}\,dx + f_{3y}\,dy + f_{3z}\,dz)\,dx \wedge dy \\
&= f_{1x}\,dx \wedge dy \wedge dz + f_{2y}\,dy \wedge dz \wedge dx + f_{3z}\,dz \wedge dx \wedge dy \\
&= (f_{1x} + f_{2y} + f_{3z})\,dx \wedge dy \wedge dz
\end{aligned}
$$

So d on 2-forms is just divergence! Overall, we can illustrate all of the operators on $\mathbb{R}^3$ in a nice chart:

$$0 \xrightarrow{\nabla} 1 \xrightarrow{\nabla\times} 2 \xrightarrow{\nabla\cdot} 3$$

Exercise 23. *Write a similar diagramme for $\mathbb{R}^4$. You can name the operators :) Hint: $0 \to 1$ and $3 \to 4$ are operators you already know.*

Exercise 24. *Prove the* ***product rule*** *for d: Let ω be a k-form on $\mathbb{R}^n$ and θ a ℓ-form on $\mathbb{R}^n$. Prove that $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta$.*

5 $d^2 = 0$

Let's recall a couple facts from multivariable calculus. Let $f(x, y, z)$ be a scalar-valued function. What is $\nabla \times (\nabla f)$?

Well, we can explicitly calculate it. We get:

$$\nabla \times (f_x, f_y, f_z) = (f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy})$$

But by Clairaut's Theorem, we know that $f_{xy} = f_{yx}, f_{yz} = f_y, f_{xz} = f_{zx}$. So $\nabla \times (\nabla f) = 0$.

Similarly, if $g(x, y, z)$ is a vector field, what is $\nabla \cdot (\nabla \times g)$? Well, $\nabla \times g = (g_{3y} - g_{2z}, g_{1z} - g_{3x}, g_{2x} - g_{1y})$. Then

$$\begin{aligned}
\nabla \cdot (\nabla \times g) &= \nabla \cdot (g_{3y} - g_{2z}, g_{1z} - g_{3x}, g_{2x} - g_{1y}) \\
&= (g_{3yx} - g_{2zx}) + (g_{1zy} - g_{3xy}) + (g_{2xz} - g_{1yz}) \\
&= g_{3xy} - g_{2xz} + g_{1yz} - g_{3xy} + g_{2xz} - g_{1yz} \quad \text{(Clairaut's Theorem)} \\
&= (g_{3xy} - g_{3xy}) + (g_{2xz} - g_{2xz}) + (g_{1yz} - g_{1yz}) \\
&= 0
\end{aligned}$$

So once again, using Clairaut's Theorem, we see that $\nabla \cdot (\nabla \times g) = 0$.

Now, let's remember that last time, we discussed that in $\mathbb{R}^3$, we have $0 \to 1 \to 2 \to 3$, where the maps are $0 \xrightarrow{\nabla} 1 \xrightarrow{\nabla\times} 2 \xrightarrow{\nabla\cdot} 3$. So in the language of differential forms, we showed that when f is a 0-form and g is a 1-form, we have $d(df) = 0$ and $d(dg) = 0$. (Why is this? Explain this to yourself.) Similarly, if we have ω a 2-form, then we must have $d(d\omega) = 0$, since $d\omega$ is a 3-form and $d(d\omega)$ is a 4-form which must be 0. And again if θ is a 3-form, then $d(d\theta) = 0$. So overall, if we have *any* form ψ on $\mathbb{R}^3$, then $d(d\psi) = 0$.

Theorem 4. *For any differential form ω on $\mathbb{R}^n$, $d(d\omega) = 0$.*

Proof. Write $\omega = \sum f_I \, dx_I$. Then

$$\begin{aligned}
d(d\omega) &= d\left(d\left(\sum f_I \, dx_I\right)\right) && \text{(definition of } \omega) \\
&= d\left(\sum_I \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} \, dx_i \wedge dx_I\right) && \text{(calculation of } d) \\
&= \sum_I \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f_I}{\partial x_i x_j} \, dx_j \wedge dx_i \wedge dx_I && \text{(calculation of } d)
\end{aligned}$$

Now, notice that both i and j go from 1 to n. So for any pair (i, j), we also sum over (j, i). So let's combine those terms. For simplicity I am just writing ? for the summation condition – it is over pairs (i, j) with $1 \le i, j \le n$, where we treat (i, j) and (j, i) as the same. So we can write $1 \le i \le j \le n$ if we want.

$$
\begin{aligned}
d(d\omega) &= \sum_I \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f_I}{\partial x_i x_j}\, dx_j \wedge dx_i \wedge dx_I \\
&= \sum_I \sum_? \frac{\partial^2 f_I}{\partial x_i x_j}\, dx_j \wedge dx_i \wedge dx_I + \frac{\partial^2 f_I}{\partial x_j x_i}\, dx_i \wedge dx_j \wedge dx_I \\
&\hspace{20em} \text{(combining terms)} \\
&= \sum_I \sum_? \left(\frac{\partial^2 f_I}{\partial x_i x_j} - \frac{\partial^2 f_I}{\partial x_j x_i} \right) dx_j \wedge dx_i \wedge dx_I \\
&\hspace{14em} (\text{since } dx_j \wedge dx_i = -dx_i \wedge dx_j) \\
&= \sum_I \sum_? 0 \qquad \text{(Clairaut's Theorem)} \\
&= 0
\end{aligned}
$$

□

So overall, we can write "$d^2 = 0$." Now that we proved that $d^2 = 0$, a natural question comes up: if $d\omega = 0$, then do we have $\omega = d\theta$?

Exercise 25. $*$ *Find an example of a* 1*-form* ω *on* $\mathbb{R}^2 \setminus \{0\}$ *where* $d\omega = 0$ *but there is no form* θ *with* $\omega = d\theta$*. Can we find an* ω *defined on all of* $\mathbb{R}^2$*?*

The answer in general turns out to be very difficult! We will start talking about this next week. It turns out that this question is related, somehow, to *counting holes* in spaces. For example, $\mathbb{R}^2 \setminus \{0\}$ has a hole (at 0), so this is possible, but $\mathbb{R}^2$ has no holes so it is impossible.

This is related to conservative vector fields: you may remember from multivariable calculus that on a simply-connected domain (a domain without holes), $F(x, y)$ is conservative if and only if $F_{2x} - F_{1y} = 0$. But if we treat $F(x, y)$ as a 1-form, $\phi = F_1(x, y)\, dx + F_2(x, y)\, dy$, then $d\phi = F_{1y}\, dy \wedge dx + F_{2x}\, dx \wedge dy = F_{2x} - F_{1y}\, dx \wedge dy$. So $F(x, y)$ is conservative if and only if $d\phi = 0$. But a conservative vector field F admits a potential function f with $\nabla f = F$, and in differential forms, we see that if F is conservative, then $\phi = df$ for some function f. So overall, $d\phi = 0$ if and only if $\phi = df$. The fact our domain was simply-connected was crucial, and soon we will see a bit of why!

Exercise 26. *Extend the previous argument to* $\mathbb{R}^3$*: prove that if* ω *is a* 1*-form on* $\mathbb{R}^3$ *with* $d\omega = 0$*, then* $\omega = df$ *for some scalar-valued function* f*.*

6 Integration

6.1 Integrals

So far we've talked a lot about forms and their constructions, but not a lot about *why* we care. We showed that differential forms encapsulate many of the tools we know from multivariable calculus, but so far, we haven't said anything about arguably the most widely used tools in multivariable calculus: integration theorems. Today we will define integration of forms, and next time we will be able to prove an incredible theorem called the Generalised Stokes's Theorem, which generalises the Fundamental Theorem of Calculus, Green's Theorem, the Fundamental Theorem of Line Integrals, Stokes's Theorem, and Gauss's Theorem (the Divergence Theorem) all at the same time. The biggest benefit of differential forms is that they allow us to take a so-called *coordinate-free* approach to integration. We will see what this means momentarily.

Let ω be a n-form on $\mathbb{R}^n$ and $U \subset \mathbb{R}^n$ an open set. Let $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$. Then we define

$$\int_U \omega := \int_U f\, dx_1\, dx_2\, \ldots\, dx_n$$

So far, this might be exactly what you'd expect.

Exercise 27. *There is a subtle problem with our definition! We know by Fubini's Theorem that $\int_K f\, dx_1\, dx_2\, \ldots\, dx_n = \int_K f\, dx_2\, dx_1\, \ldots\, dx_n$ (swapping the order of integration between x_1 and x_2), but $dx_1 \wedge dx_2 = -dx_2 \wedge dx_1$!*

Let σ be a permutation of $1, \ldots, n$. (Recall that a permutation of $\{1, \ldots, n\}$ is just a bijective function $\{1, \ldots, n\} \to \{1, \ldots, n\}$.) Then let $\omega = f\, dx_{\sigma(1)}\, dx_{\sigma(2)}\, \ldots\, dx_{\sigma(n)}$, which is just our original ω but with the dx terms in a different order. Then

$$\int_K \omega = (-1)^p \int_K f\, dx_{\sigma(1)}\, dx_{\sigma(2)}\, \ldots\, dx_{\sigma(n)}$$

Calculate what p must be (in terms of σ) to make this true.

Our definition so far doesn't let us integrate, say, over the sphere in $\mathbb{R}^3$, since the sphere is 2-dimensional. If we integrate any 3-form over the sphere in $\mathbb{R}^3$, the integral would just be 0 no matter what we integrate. So instead, we would like to integrate a 2-form over the sphere, but the sphere can't be placed into $\mathbb{R}^2$, so we don't know how to do this yet. The solution is, surprisingly, a change of variables. First, we will define the change of variables for a n-form on $\mathbb{R}^n$, then we will show how it can be extended to allow integration of all forms.

6.2 Pullbacks

In order to define such a change-of-variables, let's look at some examples we know from single-variable calculus. Let $\phi(x) : \mathbb{R} \to \mathbb{R}$ be given by $\phi(x) = 2x$. If we want to say $u = 2x$ (a u-substitution), we would write $u = \phi(x), du = d\phi(x) = \phi'(x)dx$. Then, as we know from calculus,

$$\int f(\phi(x))\,\phi'(x)dx = \int f(u)\,du$$

So this motivates our next definition: let ω be a 1-form – for simplicity, write $\omega = f(x)\,dx$. Then $\phi^*\omega := f(\phi(x))\,\phi'(x)\,dx$. If we assume that $\phi^*(\alpha \wedge \beta) = \phi^*\alpha \wedge \phi^*\beta$ and $\phi^*(\alpha + \beta) = \phi^*\alpha + \phi^*\beta$, then this is enough to define the pullback of any k-form!

Formally, we can write out this definition.

Definition 6 (pullback)**.** *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism: a smooth function that has a smooth inverse. Then let ω be a k-form. Then we define the **pullback** of ω by ϕ, denoted $\phi^*\omega$, as*

$$\phi^*\omega := \sum_I (f_I \circ \phi) d\phi_I$$

where we write $d\phi_I := d(\phi_{i_1}) \wedge d(\phi_{i_2}) \wedge \cdots \wedge d(\phi_{i_n})$.

Example 8. *Let $\phi(t) = (t^2, t-4)$ and $\omega = x\,dy - y\,dx$. We can directly calculate that $\phi^*\omega = t^2\,dt - (t-4)\,2t\,dt = 8t - t^2\,dt$.*

Let's recall how to compute a line integral. Let γ be the path given by $\gamma(t) = (t^2, t-4)$ for $0 \leq t \leq 1$. Then

$$\begin{aligned} \int_\gamma x\,dy - y\,dx &= \int_\gamma (-y, x) \cdot d\vec{r} \\ &= \int_{t=0}^{1} ((-y, x) \circ \gamma(t)) \cdot \gamma'(t)\,dt \\ &= \int_{t=0}^{1} (4 - t, t^2) \cdot (2t, 1)\,dt \\ &= \int_{t=0}^{1} 8t - t^2\,dt \end{aligned}$$

So when we calculated a line integral, we actually used the fact that $\displaystyle\int_\gamma \omega = \int_0^1 \gamma^*\omega$. So our pullback really is a change-of-variables! (And in fact line integrals are changes-of-variables from 2 variables to 1, just like a surface integral is a change of variables from 3 to 2, etc.)

Exercise 28. *Let $S(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$, and $\omega = x\,dy \wedge dz$. Compute $S^*\omega$, and explain your answer in terms of a surface integral.*

Example 9. *Let $\omega = 1\,dx\,dy$ and $\phi(x, y) = \left(\sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right)\right)$. Then*

$$
\begin{aligned}
\phi^*\omega &= (1 \circ \phi) d\phi_{1,2} \\
&= 1\, d\phi_1 \wedge d\phi_2 \\
&= 1 \left(\frac{\partial \phi_1}{\partial x}\, dx + \frac{\partial \phi_1}{\partial y}\, dy \right) \wedge \left(\frac{\partial \phi_2}{\partial x}\, dx + \frac{\partial \phi_2}{\partial y}\, dy \right) \\
&= \left(\frac{x}{\sqrt{x^2+y^2}}\, dx + \frac{y}{\sqrt{x^2+y^2}}\, dy \right) \wedge \left(\frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy \right) \\
&= \frac{x}{\sqrt{x^2+y^2}}\, dx \wedge \frac{-y}{x^2+y^2}\, dx + \frac{x}{\sqrt{x^2+y^2}}\, dx \wedge \frac{x}{x^2+y^2}\, dy \\
&+ \frac{y}{\sqrt{x^2+y^2}}\, dy \wedge \frac{-y}{x^2+y^2}\, dx + \frac{y}{\sqrt{x^2+y^2}}\, dy \wedge \frac{x}{x^2+y^2}\, dy \\
&= \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{x}{x^2+y^2}\, dx \wedge dy + \frac{y}{\sqrt{x^2+y^2}} \cdot \frac{-y}{x^2+y^2}\, dy \wedge dx \\
&= \frac{x^2}{(x^2+y^2)^{3/2}}\, dx \wedge dy - \frac{y^2}{(x^2+y^2)^{3/2}}\, dy \wedge dx \\
&= \frac{x^2+y^2}{(x^2+y^2)^{3/2}}\, dx \wedge dy \\
&= \frac{1}{\sqrt{x^2+y^2}}\, dx \wedge dy
\end{aligned}
$$

If we write $(r, \theta) = \phi(x, y)$ as our new coordinates, then we can define $dr \wedge d\theta = \phi^*(dx \wedge dy)$, so we see that $dr \wedge d\theta = \frac{1}{r}\, dx \wedge dy$, or that $r\, dr \wedge d\theta = dx \wedge dy$.

Exercise 29. *Fun fact! We just implicitly showed a very important fact (in $\mathbb{R}^2$): if we have a function $f : \mathbb{R}^n \to \mathbb{R}^n$, then $df_{1,2,\dots,n} = df_1 \wedge df_2 \wedge \cdots \wedge df_n = \det \mathrm{Jac}(f)\, dx_1 \wedge \cdots \wedge dx_n$, where $\mathrm{Jac}(f)$ is the Jacobian matrix of f, $\mathrm{Jac}(f)_{ij} = \frac{\partial f_i}{\partial x_j}$. Prove this fact.*

Exercise 30. *Let $\omega = x\, dx + y^2\, dy$ and $\theta = z\, dx \wedge dy + x\, dy \wedge dz + y\, dz \wedge dx$, and $\phi(x, y) = (x + y, x - y)$ and $\psi(x, y, z) = (x \sin y \cos z, x \sin y \sin z, x \cos y)$. Find $\phi^*\omega$ and $\psi^*\theta$.*

Exercise 31. *Let ω, η be k-forms on $\mathbb{R}^n$ and $c \in \mathbb{R}$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Using the formal definition of the pushforward we gave, prove the following.*

1. $f^*(c\omega) = cf^*\omega$
2. $f^*(\omega + \eta) = f^*\omega + f^*\eta$
3. $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$
4. $f^*(d\omega) = d(f^*\omega)$

Exercise 32. * *Prove the **change-of-variable theorem**: Let ω be a n-form on $\mathbb{R}^n$, $U, V \subset \mathbb{R}^n$ open, and $\phi : U \to V$ a (orientation-preserving) diffeomorphism. Then*

$$\int_V \omega = \int_U \phi^* \omega.$$

6.3 Chains

Now we need one more important definition. A **k-chain of length p** in $\mathbb{R}^n$ is a collection of diffeomorphisms $\{\phi_i\}_{i=1}^p : D^k \to \mathbb{R}^n$[7]. We write D^k for the open disk in $\mathbb{R}^k$ – we could also choose the open unit cube, the open unit simplex, etc. Any shape that we can easily integrate over is fine.

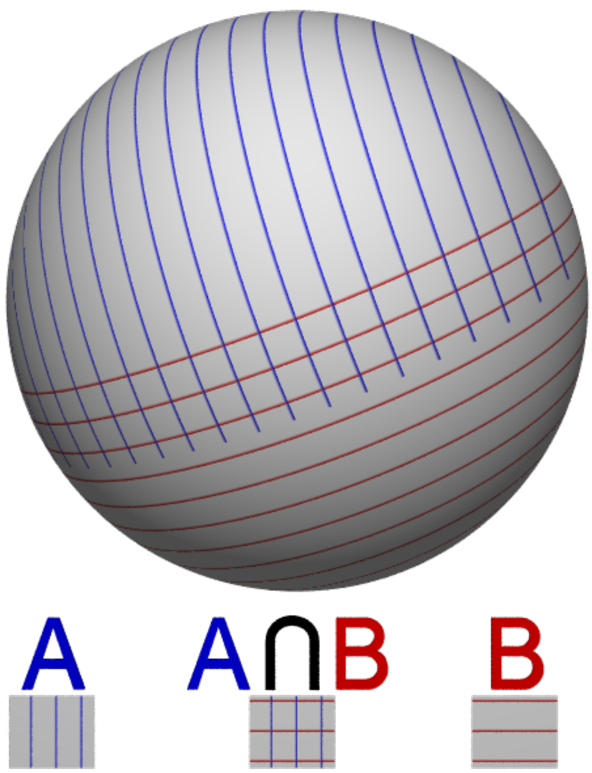

Photo Credit: RobHar on Wikimedia Commons.

[7]We need a condition on the intersections, but we will ignore this for now. This is a key part of the definition of a *manifold* which we are trying to avoid.

Example 10. *The unit sphere in* $\mathbb{R}^3$ *is a* 2*-chain of length* 2*. In the above, we can pick a diffeomorphism* ϕ_1 *that takes the unit disk in* $\mathbb{R}^2$ *to the region marked by* A*, and a diffeomorphism* ϕ_2 *that takes the unit disk in* $\mathbb{R}^2$ *to the region marked by* B*. This covers the unit sphere, making it a* 2*-chain of length* 2*.*

We can explicitly write the maps: $\phi_1(x,y) = (x, y, \sqrt{1-x^2-y^2})$ *and* $\phi_2(x,y) = (x, y, -\sqrt{1-x^2-y^2})$*. Keep in mind that the top and bottom "hemispheres" are oriented in the* ***opposite*** *direction.*

Now, we can finally define our integral. Let $C = \{\phi_i\}_{i=1}^p$ be a k-chain of length p in $\mathbb{R}^n$, and ω a k-form on $\mathbb{R}^n$. Then we define

$$\int_C \omega := \sum_{i=1}^{p} \int_{D^k} \phi_i^* \omega.$$

Example 11. *Let's integrate* $\omega = y\,dx \wedge dz$ *over the unit sphere. We already discussed that the unit sphere is a* 2*-chain of length* 2*, so*

$$\begin{aligned}
\int_{S^2} \omega &= \int_{D^2} \phi_1^*\omega + \int_{D^2} \phi_2^*\omega \\
&= \int_{D^2} (y \circ \phi_1) \cdot (d(\phi_1)_1 \wedge d(\phi_1)_3) + \int_{D^2} (y \circ \phi_2) \cdot (d(\phi_2)_1 \wedge d(\phi_2)_3) \\
&= \int_{D^2} y \cdot \left((1\,dx + 0\,dy + 0\,dz) \wedge \left(-\frac{x}{\sqrt{1-x^2-y^2}}\,dx - \frac{y}{\sqrt{1-x^2-y^2}}\,dy + 0\,dz \right) \right) \\
&\quad - \int_{D^2} y \cdot \left((1\,dx + 0\,dy + 0\,dz) \wedge \left(\frac{x}{\sqrt{1-x^2-y^2}}\,dx + \frac{y}{\sqrt{1-x^2-y^2}}\,dy + 0\,dz \right) \right) \\
&\qquad \text{(The minus sign is because of the difference in orientation.)} \\
&= \int_{D^2} -2\frac{y^2}{\sqrt{1-x^2-y^2}}\,dx\,dy \\
&= -2\int_{D^2} \frac{y^2}{\sqrt{1-x^2-y^2}}\,dx\,dy \\
&= -2\iint_{D^2} \frac{y^2}{\sqrt{1-x^2-y^2}}\,dx\,dy \\
&= -2\int_{\theta=0}^{2\pi}\int_{r=0}^{1} \frac{(r\sin\theta)^2}{\sqrt{1-r^2}}\,r\,dr\,d\theta \\
&= -2\int_{\theta=0}^{2\pi} \sin^2\theta \int_{r=0}^{1} \frac{r^3}{\sqrt{1-r^2}}\,dr\,d\theta
\end{aligned}$$

This we can directly integrate (let $u = r^2$*), and we get* $\boxed{-\frac{4\pi}{3}}$.

Exercise 33. *Calculate* $\iint_{S^2} (0, y, 0) \cdot d\vec{S}$*. Does this value match the answer we just calculated? Why or why not?*

Exercise 34. *Let the unit (hollow) cube* C *in* $\mathbb{R}^3$ *be a 2-chain of length* 6 *(one for each side). Using the above method, compute*

$$\int_C x\,dy \wedge dz$$

7 Stokes's Theorem

One of the most important and useful tools in calculus is integration theorems. You know a lot of them: Fundamental Theorem of Calculus, Fundamental Theorem of Line Integrals, Green's Theorem, Stokes's Theorem, Divergence (Gauss's) Theorem. But based on what we've seen so far, they can actually all be rephrased into one theorem: the Generalised Stokes's Theorem (I am not sure why it is just called Stokes's). It is also sometimes called the Stokes-Cartan Theorem.

Let's think about Stokes's Theorem (the one you know) for a moment. Let S be a surface in $\mathbb{R}^3$ and $\vec{F}$ a vector field. Then Stokes's Theorem says that

$$\iint_S \nabla \times \vec{F} = \oint_{\partial S} \vec{F}$$

Now, let V be a volume in $\mathbb{R}^3$. The Divergence Theorem tells us that

$$\iiint_V \nabla \cdot \vec{F} = \iint_{\partial V} \vec{F}$$

These theorems look quite similar. We have previously mentioned two facts that will help us make sense of these results: first, we can only integrate a k-form on a k-dimensional surface. Second, curl is just d one one-forms and divergence is just d on two-forms. So together, we can rewrite these results: let ω be a 1-form on $\mathbb{R}^n$ corresponding to $\vec{F}$ and θ a 2-form corresponding to $*\vec{F}$. Then

$$\int_S d\omega = \int_{\partial S} \omega$$

$$\int_V d\theta = \int_{\partial S} \theta$$

Notice that these equations are exactly the same now! And we can wrap them up into one theorem:

Theorem 5 (Stokes-Cartan Theorem)**.** *Let $S \subset \mathbb{R}^n$ be a k-dimensional surface with boundary ∂S, and ω a $(k-1)$-form on $\mathbb{R}^n$. Then*

$$\int_S d\omega = \int_{\partial S} \omega$$

The proof of this theorem is a little bit technical, but we can sketch the basic idea.

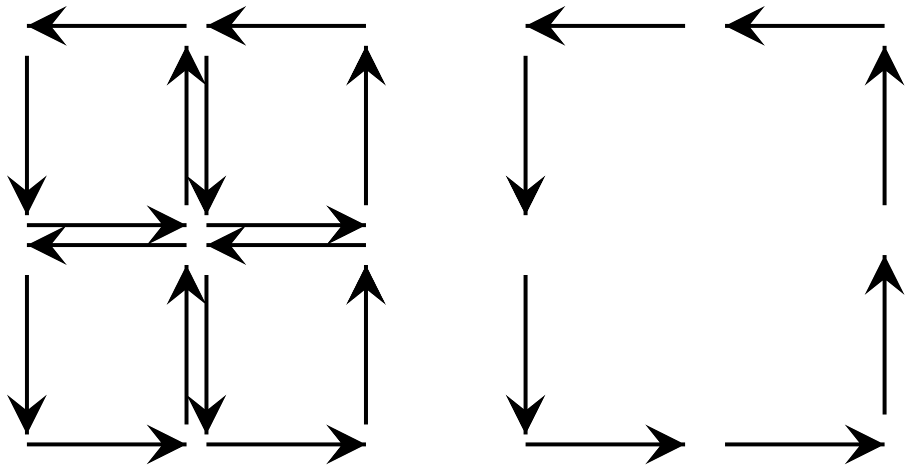

Photo credit: Krishnavedala.

The above diagramme is for 2 dimensions, but we can consider it in higher dimensions with a suitable tiling. The idea is that integrating over a region can be reduced to the boundary by creating a tiling. Each internal edge of the squares is gone over once in each direction, which will cancel out. So we can say that integrating our form over any tiling of squares is the same as just integrating it over the boundary. And as the squares get smaller and smaller, the value from going around each square approaches the derivative of ω at that point. So integrating $d\omega$ over the entire region should equal integrating ω over just the boundary.

Exercise 35. *For each of the calculus theorems (Fundamental Theorem of Calculus, Fundamental Theorem of Line Integrals, Green's Theorem, Stokes's Theorem, Divergence (Gauss's) Theorem), rewrite them in terms of the Stokes-Cartan Theorem with an appropriate choice of form. Recall that for a directed line ℓ with endpoints ℓ_0 and ℓ_1, $\partial\ell = \ell_1 - \ell_0$.*

8 Cohomology (Santiago (Go) Gonzalez)

8.1 Motivations of (Co)homology

Surprisingly, the language of differential forms is not only confined to the study of calculus. Instead, it finds a place in the study of topology and the equivalence of "shapes".

The homology and cohomology of a topological structure provides mathematicians a way to number and identify the number of holes that such a structure has. In fact, these tools help shed some light as to why a coffee mug and a donut are "the same", at least according to mathematicians. Now, we will shift our focus to defining the de Rham cohomology in terms of closed differential forms and exact differential forms.

Definition 7 (closed form)**.** *A differential form ω is closed if $d\omega = 0$.*

Example 12. *$ydx + xdy$ is closed: $d(y\,dx + x\,dy) = 1\,dydx + 1\,dxdy = 0$.*

Definition 8 (exact form)**.** *A differential form ω is exact if there exists a $k-1$ form θ such that $\omega = d\theta$.*

Example 13. *Consider $\omega = 2x\,dx + 2y\,dy$. It is easy to verify that ω is exact by considering $\theta = x^2 + y^2$: $d\theta = 2x\,dx + 2y\,dy$.*

Lemma 1. *A quick lemma coming from Definition 5 is that all exact forms must also be closed (Can you see why? Hint: go back to Chapter 5).*

8.2 Defining the de Rham Cohomology

While the following can be defined more generally on manifolds with only a little more extra legwork, we will only be caring about defining the de Rham cohomology group in n-dimensional Euclidean space (despite its name, the de Rham cohomology group is actually a vector space over $\mathbb{R}$[8]).

Take a look back at Exercise 15 now and look at what the question is asking. We already know that all exact forms are closed, but the exercise asks us whether or not all closed forms are exact. The answer is no, and the de Rham cohomology gives us a way to classify these without having to guess and check every time.

Now, for our last two definitions before the main event. For both of these, assume that we are working in $\mathbb{R}^k$ and let $p \in \mathbb{Z}_+$.

Definition 9. *$Z^p = Ker(d : \Omega^p(\mathbb{R}^k) \to \Omega^{p+1}(\mathbb{R}^k) = \{closed\ p\ forms\ on\ \mathbb{R}^k\}$*

Definition 10. *$B^p = Im(d : \Omega^{p-1}(\mathbb{R}^k) \to \Omega^p(\mathbb{R}^k) = \{exact\ p\ forms\ on\ \mathbb{R}^k\}$*

[8]Comment from Max: vector spaces are abelian groups :)

In case the definitions seem to come out of nowhere, remember that the kernel describes all elements which map to 0 (or closed forms), and the image of $p-1$ forms to p forms is the set of exact p forms). Finally, note that since every exact form is closed, we have that $B^p(\mathbb{R}^k) \subseteq Z^p(\mathbb{R}^k)$

Now, it is finally time for the definition of the de Rham cohomology:

Definition 11 (de Rham Cohomology Group of Degree p)**.** *We define the de Rham Cohomology Group of Degree p to be the quotient vector space* $H^p_{dr}(\mathbb{R}^k) = \frac{Z^p(\mathbb{R}^k)}{B^p(\mathbb{R}^k)}$. *We say that* $H^p(\mathbb{R}^k) = 0$ *for* $p < 0$ *or* $p > \dim(\mathbb{R}^k)$ *as* $\Omega^p(\mathbb{R}^k) = 0$ *for those cases. Otherwise, we say that* $H^p(\mathbb{R}^k) = 0$ *if and only if* $Z^p(\mathbb{R}^k) = B^p(\mathbb{R}^k)$.

Before we get to the fun part of counting holes in space, there is a bunch more background that is needed. We will forsake this background and instead present it in a swift manner without proof. The theorems (with proof) may be found in Chapter 17 of Lee's *Intro to Smooth Manifolds.*

Theorem 6. *If M and N are homotopy equivalent smooth manifolds with or without boundary, then* $H^p(M) \cong H^p(N)$ *for each p. The isomorphisms are induced by any smooth homotopy equivalence* $F : M \to N$.

Corollary 1. *The de Rham cohomology groups are topological invariants: if M and N are homeomorphic smooth manifolds with or without boundary, then their de Rham cohomology groups are isomorphic*

Theorem 7. *For* $n \geq 1$*, the de Rham cohomology groups of* S^n *are*
$$H^p(S^n) \cong \begin{cases} \mathbb{R} & \text{if } p = 0, n \\ 0 & \text{if } 0 < p < n \end{cases}$$

Ignoring the jargon about manifolds, what the above basically tells us is that the cohomology of topological structures which are homeomorphic (i.e. two objects that can be transformed into each other without "ripping") is the same! They also tell us the same for objects which are homotopy equivalent (for now just think of this as a weaker notion of homeomorphic). Now, we are ready to calculate holes mathematically!

8.3 Calculating Cohomology of $\mathbb{R}^2 \setminus \{(0,0)\}$

Example 14. *Calculate* $H^1(\mathbb{R}^2 \setminus \{(0,0)\})$

First, note that there exists $\mathbb{R}^2 \setminus \{(0,0)\}$ is homotopy equivalent to S^1, the circle. The proof of this (along with the rigorous definition of what it means to be homotopy equivalent) will be left as an exercise for the reader. For now, just think that the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$ has exactly one hole (the origin) while S^1 the circle also has one hole. So, by the above theorem, we see that $H^1(\mathbb{R}^2 \setminus \{(0,0)\}) \cong H^1(S^1) \cong \mathbb{R}$. Well that was anti-climactic... Unfortunatley, most of the legwork behind proving the cohomology of more interesting spaces is far outside of the reach of this directed reading project. However, if the reader is curious to look at more, Lee's *Intro to Smooth Manifolds* provides some prime examples along wit rigorous proof for everything covered here in Section 8.

Definition 12. *A homotopy between two continuous functions f and g from topological spaces X and Y is defined to be a continuous function $H : X \times [0,1] \to Y$ such that for all $x \in X$ $H(x,0) = f(x), H(x,1) = g(x)$.*

Definition 13. *Two topological spaces X and Y are said to be homotopy equivalent if there exists a pair of continuous maps $f : X \to Y$, $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y*

Exercise 36. *Show that $\mathbb{R}^2 \setminus \{(0,0)\}$ is homotopy equivalent to S^1*

9 Application to Liouville's Theorem of Hamiltonian Mechanics (Nick Wojnar)

9.1 Introduction to the Hamiltonian

9.1.1 Defining the Hamiltonian

Consider a system with generalized coordinates $\{\mathbf{q}, \dot{\mathbf{q}}\}$. Given the kinetic energy T and the potential function V, we can write the Lagrangian of the system.

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathrm{T} - \mathrm{V}$$

Recall that we define the generalized momenta p_i of the system as follows:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

We now define the Hamiltonian $\mathcal{H}$. Note that we change our coordinates from $\{\mathbf{q}, \dot{\mathbf{q}}\} \to \{\mathbf{q}, \mathbf{p}\}$

$$\mathcal{H}(\mathbf{q}, \mathbf{p}, t) = \sum_i p_i \dot{q}_i - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$$

We can demonstrate that this quantity satisfies Hamilton's Equations.

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial p_i} &= \dot{q}_i(\mathbf{q}, \mathbf{p}, t) \\ \frac{\partial \mathcal{H}}{\partial q_i} &= -\dot{p}_i \\ \frac{\partial \mathcal{H}}{\partial t} &= \frac{d\mathcal{H}}{dt} \end{aligned}$$

9.1.2 Phase-Space

While the Lagrangian describes the evolution of generalized coordinates over time, the Hamiltonian's equations describe the time-evolution of a different phase-space with coordinates $\{\mathbf{q}, \mathbf{p}\}$, which we can then transform into the equations of motion of the system, i.e. we can explicitly solve for $\mathbf{q}(t)$. This formulation offers interesting geometric features, including Lioville's Theorem.

Consider a simple mass-spring system with generalized coordinates $\{\mathbf{x}, \dot{\mathbf{x}}\}$, where $m = k = 1$. We can find the Lagrangian and generalized momentum of the system to define the Hamiltonian.

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 \\ \mathcal{H}(\mathbf{q}, \mathbf{p}, t) &= \sum_i p_i \dot{q}_i - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \\ &= p\dot{x} - (\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2) \\ &= \frac{1}{2}(p^2 + x^2) \end{aligned}$$

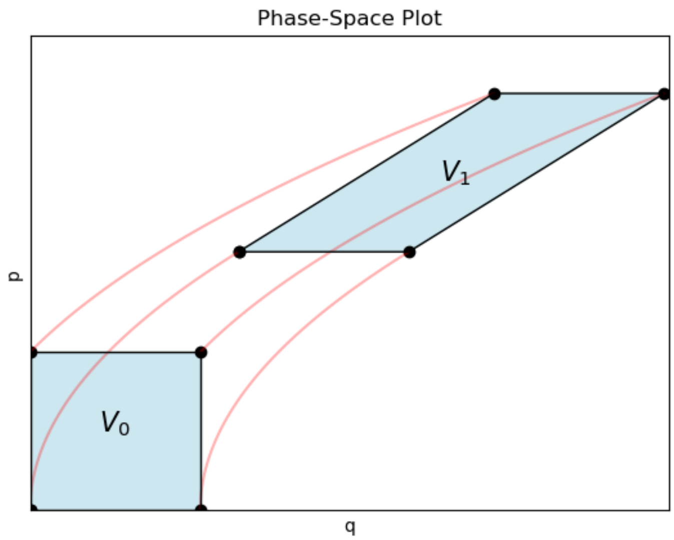

Figure 1: Phase-Space Plot for a series of 4 particles with initial position and/or momentum. This specific evolution arises due to free particles in gravity.

Since this $\mathcal{H}$ is not explicitly time-dependent, it can be treated as a constant with respect to time. As such, we can plot a phase-space diagram of this system. As we can tell, it will be a circle of radius $\sqrt{2\mathcal{H}}$, and we can use the Hamiltonian equations to determine that the circle is oriented clockwise.

Exercise 37. *Find the Hamiltonian for a mass m at the end of a simple pendulum of length l that makes an angle θ with the vertical. Take the top of the pendulum to be the origin.*

9.2 Liouville's Theorem

9.2.1 The Statement

Consider a closed region $\mathcal{S}$ in a $2n$-dimensional phase space, i.e. a space with n particles with position and momentum in one space dimension. Call the volume of the enclosed region $\mathcal{V}$, and allow the region to move through phase space according to Hamilton's equations. Liouville's Theorem states that the total volume enclosed in our region will remain constant over time, namely that

$$\frac{d\mathcal{V}}{dt} = 0$$

Figure 1 demonstrates the evolution of a region $\mathcal{S}$ from $t = 0$ to $t = 1$, with a 2-volume (area) of V_0 to V_1. One can show numerically that $V_0 = V_1$ given the exact equations of motion. We can show that this will hold for any region with any Hamiltonian.

9.2.2 Proof

We define the $(2n-1)$-form current flow through phase space $\mathbf{v}$ as the time derivative of our position in phase space.

$$\mathbf{v} = \frac{d}{dt}(\mathbf{q}, \mathbf{p})$$
$$\mathbf{v} = (\sum_{i=1}^{n} \dot{q}_i \cdot d\mathrm{Q}_i + \dot{p}_i \cdot d\mathrm{P}_i)$$

Where I define:

$$d\mathrm{Q}_i = dq_1 \wedge ... \wedge dq_{i-1} \wedge dq_{i+1} \wedge ... \wedge dq_n \wedge p_1 \wedge ... \wedge p_n$$
$$d\mathrm{P}_i = dq_1 \wedge ... \wedge dq_n \wedge p_1 \wedge ... \wedge dp_{i-1} \wedge dp_{i+1} \wedge ... \wedge p_n$$

The change in volume over time can be pictures as small changes along the boundary according to the current flow. That is:

$$\frac{d\mathcal{V}}{dt} = \oint_{\partial\mathcal{S}} \mathbf{v} \cdot d\mathrm{s}$$

And written as a differential form:

$$\frac{d\mathcal{V}}{dt} = \int_{\partial\mathcal{S}} \mathbf{v}$$

By Stoke's Theorem:

$$\frac{d\mathcal{V}}{dt} = \int_{\mathcal{S}} d\mathbf{v}$$
$$= \int_{\mathcal{S}} \sum_{i=1}^{n} (\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}) dq_1 \wedge ... \wedge dq_n \wedge p_1 \wedge ... \wedge p_n$$

This is simply the divergence of $\mathbf{v}$. By Hamilton's equations, we can find:

$$\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} = \frac{\partial}{\partial q_i}(\frac{\partial \mathcal{H}}{\partial p_i}) + \frac{\partial}{\partial p_i}(-\frac{\partial \mathcal{H}}{\partial q_i})$$
$$= \frac{\partial^2 \mathcal{H}}{\partial q_i \partial p_i} - \frac{\partial^2 \mathcal{H}}{\partial p_i \partial q_i}$$
$$= 0$$

Therefore, we achieve our result,

$$\frac{d\mathcal{V}}{dt} = \int_{\mathcal{S}} \sum_{i=1}^{n} (0) dq_1 \wedge ... \wedge dq_n \wedge p_1 \wedge ... \wedge p_n$$
$$= \int_{\mathcal{S}} 0 dq_1 \wedge ... \wedge dq_n \wedge p_1 \wedge ... \wedge p_n$$
$$= 0$$

thus proving Liouville's Theorem.

Exercise 38. *Verify that Liouville's Theorem holds for particles in free-fall, as represented in Figure 1.*

10 Solutions

10.1 Exercise 11 (Max)

It is just the Kronecker product $v \otimes w = \begin{pmatrix} ev & fv \\ gv & hv \end{pmatrix} = \begin{pmatrix} ae & be & af & bf \\ ce & de & cf & df \\ ag & bg & ah & bh \\ cg & dg & ch & dh \end{pmatrix}$.

10.2 Exercise 18 (Nick)

Assume that there exists some $\omega' = \omega$ such that $\omega' = \sum_{v_i} g_i(\mathbf{x})v_i$.

$$\begin{aligned} \omega &= \omega' \\ \omega - \omega' &= \mathbf{0} \\ \sum_{v_i} f_i(\mathbf{x})v_i - \sum_{v_i} g_i(\mathbf{x})v_i &= \mathbf{0} \\ \sum_{v_i} (f_i - g_i)(\mathbf{x})v_i &= \mathbf{0} \\ \implies f_i - g_i &= 0 \\ f_i &= g_i \end{aligned}$$

Since $\{v_i\}$ is a basis, each v_i is linearly independent, so the relation $f_i - g_i = 0$ must hold for all i. This means that $f_i = g_i$ for all i, so ω and ω' must be the same.

10.3 Exercise 9 (Nick)

I will omit the wedges for conciseness. Consider θ a 1-form on $\mathbb{R}^3$ spanned by $\{x, y, z\}$. The most general θ is as follows:

$$\theta = f_1(x, y, z)dx + f_2(x, y, z)dy + f_3(x, y, z)dz$$

One example of ϕ a 2-form:

$$\phi = g_{1,2}dxdy + g_{2,3}dydz + g_{3,1}dzdx$$

One example of ψ a 3-form:

$$\psi = h_{1,2,3}dxdydz$$

10.4 Exercise 10 (Nick)

Let $\theta = f_1dx + f_2dy + f_3dz$ and $\omega = g_1dxdy + g_2dydz + g_3dzdx$.

$$\theta \wedge \omega = (f_1g_2 + f_2g_3 + f_3g_1)dxdydz$$

In this calculation, it is important to note that any repeats ($dx \wedge dxdy$) are all equal to zero.

10.5 Exercise 13 (Nick)

Consider a 0-form $\omega_0 = f(\mathbf{x})$ on $\mathbb{R}^4$. Then $d\omega_0 = \sum_i \frac{\partial f}{\partial x_i} dx_i$. This is just the gradient of f.

Consider a 1-form $\omega_1 = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 + f_4 dx_4$.

$$\begin{aligned} d\omega_1 &= \sum_i \sum_j \frac{\partial f_j}{\partial x_i}(dx_j \wedge dx_i) \\ &= \sum_{i<j} (\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}) dx_i \wedge dx_j \end{aligned}$$

This looks a lot like the curl in $\mathbb{R}^3$! For this exercise, I want to call it the 2-curl since it is a 2-form where each $dx_i dx_j$ has two terms.

Consider the 2-form $\omega_2 = \sum_{i,j} g_{i,j} dx_i dx_j$.

$$d\omega_2 = \sum_{i,j,k} (\frac{\partial g_{i,j}}{\partial x_k} - \frac{\partial g_{j,k}}{\partial x_i} + \frac{\partial g_{k,i}}{\partial x_j}) dx_i dx_j dx_k$$

I will call this the 3-curl (I know, not very creative). It's a pretty big hassle to expand this out.

Predictably, the exterior derivative of ω_3 a 3-form will just be its divergence.

10.6 Exercise 15 (Nick)

Consider $\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$. Then, $d\omega = \frac{(x^2-y^2)+(y^2-x^2)}{(x^2+y^2)^2} = 0$. We might think that $\theta = -\arctan(x/y)$; however, this is undefined for all $y = 0$. So, θ is a 0-form on $\mathbb{R}^2 \setminus \{(x, y)|y = 0\}$. This domain is different from ω :(

10.7 Exercise 26 (Go)

Define $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to S^1$ such that $f(x, y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$. Now, set $g : S^1 \to \mathbb{R}^2 \setminus \{(0, 0)\}$ where $g(x, y) = (x, y)$. Here, f is the projection map onto the circle whereas g represents the inclusion map of S^1 onto $\mathbb{R}^2 \setminus \{(0, 0)\}$. Now, it is time to check that their compositions are homotopic to id_X and id_Y.

Define $H((x, y), t) = (1-t)(f \circ g)(x, y) + t(x, y) = (1-t)(x, y) + t(x, y)$. This is because on S^1, $x^2 + y^2 = 1$, so we can simplify $f \circ g = (x, y)$. Now, we see that $H((x, y), 0) = (x, y) = f \circ g(x, y)$ and $H((x, y), 1) = (x, y)$, the identity map on S^1. Hence, $f \circ g$ is homotopic to the identity map on S^1.

Now, we check for the punctured plane. Define $G((x, y), t) = ((1 - t)x + t\frac{x}{\sqrt{x^2+y^2}}, (1 - t)y + t\frac{y}{\sqrt{x^2+y^2}})$. I leave it to the reader to verify, but we see that indeed $g \circ f$ is homotopic to the identity map on $\mathbb{R}^2 \setminus \{(0, 0)\}$. Thus, we have shown that $\mathbb{R}^2 \setminus \{(0, 0)\}$ and S^1 are homotopy equivalent!