Lectures on Critical Level

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Contents

1 Introduction and Review

Parts of this lecture follow the introduction of [FG05].

1.1 Kazhdan-Lusztig Theory

Let's start by discussing Kazhdan-Lusztig polynomials. They can be defined purely combinatorially, with no reference to representation theory or geometry.

Let W be a finite Coxeter group generated by simple reflections $\{s_i\}_{i\in I}$, and $\ell(w) : C \to \mathbb{N}$ the length function.

Definition 1 (Hecke algebra)**.** *Define $H(W)$ (or just H when W is clear) to be the $\mathbb{Z}[q^{\pm 1/2}]$-algebra generated by $\{T_w\}_{w\in W}$ with the following relations:*

1. $T_y T_w = T_{yw}$ *when* $\ell(yw) = \ell(y)\ell(w)$
2. $(T_{s_i} + 1)(T_{s_i} - q) = 0$ *for* $i \in I$

*Then H, called the **Hecke algebra** of W, admits an involution $D : H \to H$ that sends $q^{1/2} \mapsto q^{-1/2}$ and T_{s_i} to $T_{s_i}^{-1}$.*

This algebra has a canonical basis $\{T_w\}$ and another basis $\{C_y\}_{y\in W}$ such that the transition matrix is upper-triangular (with respect to the Bruhat ordering on $\{T_w\}$) and $D(C_y) = q^{-\ell(y)/2}C_y$ for $y \in W$. It turns out that this basis is unique, and we define $P_{y,w}$ to be the element of $\mathbb{Z}[q^{\pm 1/2}]$ indexed by y and w.

There is also an explicit definition of these polynomials using a recursive definition but it isn't necessary. It is a nontrivial fact, but it turns out that $P_{y,w}$ lie in $\mathbb{Z}[q] \subset \mathbb{Z}[q^{\pm 1/2}]$.

Conjecture 1 (Kazhdan-Lusztig, [KL93])**.** *If $\mathfrak{g}$ is a simple Lie algebra with Weyl group W, then*

$$\operatorname{ch}(L_w) = \sum_{y\le w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) \operatorname{ch}(\Delta_y)$$

Equivalently,

$$\operatorname{ch}(M_w) = \sum_{y\le w} P_{w_0 w, w_0 y}(1) \operatorname{ch}(L_y)$$

Where Δ_y denote Vermas and L_w denotes their simple quotient. This conjecture can be equivalently stated as the following question: "how can we determine the multiplicity of L_y in M_w for $y \le w$?"

This conjecture relates on the left, a purely algebraic object, and on the right, a purely combinatorial object. But its proof is almost entirely geometric, and launched the field of *geometric repersentation theory.*

Let G be a simple complex algebraic group, B be a Borel subgroup of G, and $\mathfrak{g}, \mathfrak{b}$ the Lie algebras. Then G/B has the structure of a projective variety, and is known as the *flag variety*. For this lecture series, whenever we implicitly use a variety, we always mean the flag variety. For example, a sheaf $\mathcal{F}$ is a sheaf on G/B, and $\Gamma(\mathcal{F}) := \Gamma(G/B, \mathcal{F})$.

Now, let us consider the category of $\mathscr{D}$-modules on G/B. If you've never heard of a $\mathscr{D}$-module, just think of them as quasicoherent sheaves with an action of $\mathscr{D}_{G/B}$, the sheaf of differential operators. We have that $U(\mathfrak{g})$ acts on $\Gamma(\mathcal{F})$ for any $\mathscr{D}$-module $\mathcal{F}$, and $Z(\mathfrak{g}) := Z(U(\mathfrak{g}))$ acts on $\Gamma(\mathcal{F})$ by a character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$. For a chosen character χ_0, denote $\mathfrak{g}_{\chi_0}$-mod for the category of $\mathfrak{g}$-modules where $Z(\mathfrak{g})$ acts by χ_0.

Then, by a celebrated result of Beilinson-Bernstein [BB81], we have

$$\Gamma : \mathscr{D}(G/B)\text{−mod} \to \mathfrak{g}_{\chi_0}\text{−mod}$$

is an equivalence of categories. In particular, we can restrict the BB functor to a functor from $\mathcal{O}$ (the BGG category) to "regular holonomic" $\mathscr{D}$-modules (whatever that means). Then, we can use the Riemann-Hilbert correspondence to get perverse sheaves on G/B. Then, our multiplicity calculation becomes a purely geometric computation of intersection cohomology on G/B, which can be done (but we will ignore for today).

Thus, we can prove KL by rephrasing it geometrically. The key point here is the "geometrisation," or localisation.

1.2 Kac-Moody Algebras

Kac-Moody algebras, developed simultaneously in 1967 by Victor Kac and Robert Moody, are an important extension of Lie algebras to infinite dimensions. In particular, in this lecture series, we will be working with *(completed) affine Kac-Moody algebras*.

Let us remain in the same setup, where G is a simple complex Lie group and $\mathfrak{g}$ its Lie algebra. Usually we define our affine Kac-Moody algebra as $\hat{g} := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}\, c$ (potentially with $\rtimes \mathbb{C}\, d$). This can also be constructed explicitly from a generalised Cartan matrix and generators and relations à la Serre, and this is typically thought of as *the* affine Kac-Moody algebra.

Instead, for this lecture series, we will work with a "completion" (see [Gan21]) of this construction, and one that is more useful geometrically. To construct it, we start by defining the *loop algebra* $\mathfrak{g}((t)) := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((t))$ (note: sometimes the loop algebra is defined as $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$, but this may depend on convention). We then take our central extension like before: we have $0 \to \kappa\, \mathbb{C} \to \hat{g}_\kappa \to \mathfrak{g}((t)) \to 0$. Such central extensions are classified by $\mathfrak{g}$-invariant bilinear forms which are in turn classified by $H^2_{\mathrm{Lie}}(\mathfrak{g}((t)), \mathbb{C})$, and a homological algebra computation that we won't do tells us that this is one-dimensional when $\mathfrak{g}$ is simple.

We can identify 1 with the Killing form. Under this classification, $\kappa = -\frac{1}{2}$ is the critical level. This is also written $\kappa = -h^\vee$ when we identify 1 with the *minimal* bilinear form, the form κ_m where $\kappa_m(\alpha, \alpha) = 2$ for α the longest root ([FG05]). Most people seem to agree that $\kappa_m = 1$, so we will use this notation. More on level later!

Let's talk geometry. We can treat $\mathbb{C}[[t]]$ as the formal disk (the formal neighbourhood of a point) – really its ring of functions, but we will abusively equate the two for a moment. And $\mathbb{C}((t))$ is the punctured formal disk. In this understanding, $\mathfrak{g}((t))$ is the formal completion of $\mathfrak{g}(t)$.

One concern that you might have is whether or not changing $\mathfrak{g}(t)$ to $\mathfrak{g}((t))$ affects the representation theory. The good news is that in most cases (including any representation in $\mathcal{O}$), we can lift from $\mathfrak{g}(t)$ to $\mathfrak{g}((t))$ (and we can always go from $\mathfrak{g}((t))$ to $\mathfrak{g}(t)$). There are some technical details here but we will ignore them and simply treat $\hat{\mathfrak{g}}_\kappa := \mathfrak{g}((t)) \oplus \mathbb{C}\,\kappa$ where κ is the level normalised to $\kappa_m = 1$.

The Kazhdan-Lusztig Conjecture was a huge milestone in the representation theory of Lie algebras. A natural question is, how can we extend it to the affine case? We will discuss this briefly later today, but the entire lecture on Monday will be about this.

1.3 Level and the Virasoro Algebra

This section largely follows a previous talk of mine, [Ste24]. When $\kappa - \kappa_{crit} \in \mathbb{Q}_{>0}$, we call κ *positive.* If $\kappa \neq \kappa_{crit}$ and κ is not positive, we call κ *negative.* In particular, if $\kappa \notin \mathbb{Q}$, κ is negative ([CDR21] 2.7.2).

We can define $\mathcal{O}_\kappa(\mathfrak{g})$ to be the equivalent of $\mathcal{O}$ for $\hat{\mathfrak{g}}_\kappa$, with a smoothness condition: $\forall v \in V, \exists n \in \mathbb{N}$ so that $t^n\,\mathfrak{g}[[t]]v = 0$[1].

There are a lot of differences between positive, negative, and critical level:

1. $Z(\mathcal{O}_\kappa(\mathfrak{g})) = \mathbb{C} \iff \kappa \neq \kappa_{crit}$
2. For $V \in \mathcal{O}_\kappa(\mathfrak{g})$ with some regularity condition I won't define, κ being negative implies V has a finite composition series and κ being irrational implies V is semisimple ([Her] 5.1).
3. (Kazhdan-Lusztig Equivalence) When κ is outside of $[-\frac{1}{2}, \infty)$, we have $\mathcal{O}_\kappa \simeq \mathrm{Rep}_q(G)$ where $q = e^{2\pi i}$, [Gan21].

In the affine case, we have the *affine Grassmannian*, defined as $G((t))/G[[t]]$, and the *affine flag scheme*, defined as $G((t))/I$, where I is an *Iwahori subgroup*: the preimage of a Borel $B \subset G$ under $G((t)) \mapsto G$. The geometry of the affine flag scheme varies based on level:

1. κ is positive: X admits a Birkhoff decomposition with pieces having codimension $\ell(w)$ (and each piece being isomorphic to $\mathbb{A}^\infty$), and is so-called "thick"
2. κ is negative: X admits a Bruhat decomposition with pieces having dimension $\ell(w)$, and is so-called "thin"[2]

[1] This is a monoidal category with the *fusion product* yet, noting that the usual tensor product adds level. It is also the heart of the canonical t-structure on $\hat{\mathfrak{g}}$-mod, [CDR21] 2.9.2.

[2] This is equivalent to the stack of G-bundles on $\mathbb{P}^1$ with a trivialisation at ∞.

3. κ is critical: X is so-called "semi-infinite" and the geometry becomes very strange. This will be discussed later on.

The geometry here is quite rich, but we will be mostly working on the affine Grassmannian itself in these lectures.

Another important object that relies on level is the Virasoro algebra. Let W be the Witt algebra: it is generated by vector fields $L_n := -z^{n+1}\partial_z$ for $n \in \mathbb{N}$, and has a Lie bracket $[L_n, L_m] = (m-n)L_{m+n}$. The Virasoro algebra V is defined as the central extension of the Witt algebra: $V = W \oplus \mathbb{C}\, c$, where c is a central charge. Just like $\hat{g}_\kappa$, we will write V_c to keep track of our central charge. V_c has Lie bracket $[c, L_n] = 0$ and $[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$.

It turns out that for some c depending on κ, $V_c \subset U(\hat{\mathfrak{g}})$, through a construction called the Sugawara construction. We will briefly describe it now. Recall that $\mathfrak{g}$ admits a basis $\{J^a\}$ where $1 \leq a \leq \dim \mathfrak{g}$. Then $\hat{\mathfrak{g}}_\kappa$ admits a basis $\{J^a_n\}$ with $1 \leq a \leq \dim \mathfrak{g}$ and $n \in \mathbb{Z}$. We can define

$$J^a := \sum_{n \in \mathbb{Z}} J^a_n$$

a formal power series, and the *affine Casimir element*

$$S(z) := \frac{1}{2} \sum_{a=1}^{\dim \mathfrak{g}} : J^a(z) J^a(z) :$$

where $: AB :$ denotes the *standard ordering* which we will omit defining (it is necessary because naïvely multiplying this does not result in a well defined expression). Then this can be written as $S(z) = \sum_{n \in Z} S_n z^{-n-2}$, and we call S_n the *Segal-Sugawara operators*[3].

One can write this explicitly: we have a map $V_c \to U(\hat{g}_\kappa)$ where

$$L_0 \mapsto \frac{1}{2(\kappa + h^\vee)} \left(\sum_{a=1}^{\dim \mathfrak{g}} \sum_{m \in \mathbb{Z}} J^a_{n-m} J^a_m \right)$$

$$L_n \mapsto \frac{1}{2(\kappa + h^\vee)} \left(2 \sum_{a=1}^{\dim \mathfrak{g}} \sum_{m \in \mathbb{Z}} J^a_{-m} J^a_m + J^0_a J^0_a \right)$$

and we see that $c = \frac{\kappa \cdot \dim \mathfrak{g}}{\kappa + h^\vee}$ for this to work. It is immediately clear here that something falls apart when $\kappa = -h^\vee$! As we will see later, it turns out that we can slightly modify the Sugawara construction at the critical level, and our resulting Segal-Sugawara operators will actually generated the centre! We will be able to compute

$$[S_n, A_m] = -\frac{\kappa - \kappa_c}{\kappa_0} n A_{n+m}$$

for $A_m \in U(\hat{\mathfrak{g}}_\kappa)$. The tools to compute this will be *vertex algebras*.

[3] Technically these lie in the completion $\widehat{U(\hat{\mathfrak{g}}_\kappa)}$ completed as $\varprojlim U(\hat{\mathfrak{g}}_\kappa)/\, \mathfrak{g} \otimes t^n\, \mathbb{C}[[t]]$.

1.4 Vertex Operator Algebras

Definition 2 (vertex algebra, [Fre07] 2.2.2)**.** *A **vertex algebra** over $\mathbb{C}$ is the data of*

1. *A vector space V (the **space of states**)*
2. *A vector $|0\rangle \in V$ (the **vacuum vector**)*
3. *An endomorphism $T : V \to V$ (the **translation operator**)*
4. *A linear map $Y(-, z) : V \to \operatorname{End} V[[z^{\pm 1}]]$ sending vectors in V to **fields**, aka **vertex operators** on V (the **state-field correspondence**)*

$$A \in V \mapsto Y(A, z) := \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$$

satisfying

1. $Y(|0\rangle, z) = \mathrm{id}_V$
2. $Y(A, z)|0\rangle = A + z(...) \in V[[z]]$
3. $[T, Y(A, z)] = \partial_z Y(A, z)$
4. $T|0\rangle = 0$
5. ***(locality)*** *For any $A, B \in V$ there is some $N \in \mathbb{N}$ so that*

$$(z - w)^N Y(A, z)Y(B, w) = (z - w)^N Y(B, w)Y(A, z).$$

Given our affine Kac-Moody algebra $\hat{\mathfrak{g}}_\kappa$, we can construct a vertex algebra

$$V_k(\mathfrak{g}) := U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[[t]]\oplus\mathbb{C}\,1)} \mathbb{C}_\kappa$$

called the **vacuum Verma module** of level κ. We will not prove anything about this vertex algebra, but it will be very important later.

Definition 3 (vertex operator algebra)**.** *Given a vertex algebra $(V, |0\rangle, T, Y)$, we say it is a **vertex operator algebra** if there is some $\omega \in V$ so that $Y(\omega, z)$ is the Virasoro field:*

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

This is sometimes called a conformal vertex algebra ([Fre07]).

When κ is not critical, the vacuum Verma of level κ is a vertex operator algebra. As we saw before, when κ is critical, the Sugawara construction fails and the Virasoro algebra collapses, so it is impossible for V_κ to be a vertex operator algebra. So once again we see that the critical level is a unique case!

In the third lecture, we will be investigating the centre of $U(\mathfrak{g})$. We will motivate and expand on this later, but for now, we will simply say that the vertex algebra construction V_κ will be critical (pun intended) in finding the centre of $U(\mathfrak{g})$.

2 The Centre

Comment: originally Lectures 2 and 3 were in the opposite order, but I think the content is more understandable and motivated this way.

2.1 Introduction

BGG $\mathcal{O}$, Blocks, and Linkage

Why do we care about the centre?

Let G be a simple complex Lie group and $\mathfrak{g}$ its Lie algebra. Recall that $\mathfrak{g}$ has a *triangular decomposition*

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

We can construct the category $\mathcal{O}$ as the subcategory of $\mathfrak{g}$-mod with the following properties:

1. M is finitely generated
2. $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ ($M_\lambda := \{v \in M | \forall h \in \mathfrak{h}, h \cdot v = \lambda(h)v\}$
3. M is locally $\mathfrak{n}$-finite: $\forall v \in M, \mathfrak{n} \cdot v$ is finite-dimensional.

Inside of $\mathcal{O}$, we have Verma modules $\Delta(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$, and we write $L(\lambda)$ for the irreducible representation of $\mathfrak{g}$ of highest weight λ, which is also the unique simple quotient of $\Delta(\lambda)$. We use $\mathrm{ch}(V)$ to denote the formal character of V, defined as $\mathrm{ch}(V) := \sum_{\mu \in \mathfrak{h}^*} e^\mu \dim_{\mathbb{C}} V^\mu$, where V^μ is the weight space of weight μ.

A priori, it is not clear that $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ is well-defined as a $\mathfrak{g}$-module. It turns out that in characteristic 0, $U(\mathfrak{g})$-mod $\cong$ $\mathfrak{g}$-mod, so we can treat $U(\mathfrak{g})$-modules as $\mathfrak{g}$-modules, and vice versa.

Let's look at how $Z(\mathfrak{g})$ acts on $\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$. Since $\mathbb{C}_\lambda$ is one-dimensional, $\Delta(\lambda)$ is genearted by $1 \otimes 1$ as a left $U(\mathfrak{g})$-module. Since $Z(\mathfrak{g})$ is central, it acts on $v_\lambda = 1 \otimes 1$ by a homomorphism $\chi_\lambda : \mathbb{Z}(\mathfrak{g}) \to \mathbb{C}$, which is known as a central character. In fact, if we take the PBW decomposition of $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (U(\mathfrak{n}^+) + U(\mathfrak{n}^-))$, we can take $\pi : U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{h})$ a projection. Then $\chi_\lambda(z) = \lambda(\pi(z))$.

Then, consider the translation $\sigma : \lambda \mapsto \lambda - \rho$, which is a $\mathbb{C}$-algebra automorphism on $S(\mathfrak{h})$. We can take $\Theta := \sigma \circ \pi : Z(\mathfrak{g}) \to S(\mathfrak{h})$, which is called the *twisted Harish-Chandra homomorphism.*

Theorem 1 (Harish-Chandra, [CW20] 8.5)**.**

1. *Θ descends to an isomorphism $Z(\mathfrak{g}) \to S(\mathfrak{h})^{(W,\bullet)} \cong \mathbb{C}[\mathfrak{h}^* /(W, \bullet)]$.*
2. *Every $\mathbb{C}$-algebra morphism $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is a χ_λ for some λ, and $\chi_\lambda = \chi_\mu \iff \mu \in W \bullet \lambda$.*

Given some $M \in \mathcal{O}$, write M^χ for the subspace $M^\chi = \{m \in M | (z - \chi(z))^n \cdot m = 0\}$ for some $n \geq 0$ depending on z. Since M is generated by finitely many weight vectors, M decomposes as a sum of finitely many nontrivial M^χ. Thus, we can write

$$\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^* / (W, \bullet)} \mathcal{O}_\lambda$$

writing $\mathcal{O}_\lambda$ as the full subcategory of modules where $M = M^{\chi_\lambda}$. So in some sense, our category "fibres" over the centre, in a way that can be made more precise geometrically (which we will discuss in the Feigin-Frenkel discussion).

2.2 Harish-Chandra Theorem

This section and the next are an extended version of the approach Ivan Loseu used to explain this to me.

We quickly define parabolic and Levi subgroups. When G is an algebraic group and $B \subset G$ a Borel subgroup, then any subgroup P with $B \subset P \subset G$ is called *parabolic*. There is a theorem of Levi that says that any parabolic subgroup P can be written as $P = L \ltimes N$ where N is the radical of P and L is a semisimple group called the *Levi factor* of P. A subgroup L is called *Levi* if it is a centraliser of a subtorus. It is a theorem that the Levi factor of a parabolic subgroup is a Levi subgroup, and every Levi occurs as the Levi factor of a parabolic.

We can construct *standard* parabolic subgroups P_I, where I is a subset of simple roots in the following way: let U_α be the matrix associated to s_α. Then $P_I = \langle T, U_{\pm alpha} | \alpha \in I \rangle$. All parabolic subgroups are conjugate to a standard form, and similar for Levis, so we will treat parabolics and Levis as classified by sets of simple roots.

Example 1. *Let $G = \mathrm{GL}_n$ and B the subgroup of upper-triangular matrices. Then any parabolic P is a subgroup of block-upper-triangular matrices, and its Levi factor is block-diagonal matrices corresponding to the same partition.*

We will now discuss the proof of the Harish-Chandra Theorem in five steps. Let $\mathfrak{g}$ be a reductive Lie algebra, $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subgroup, containing $\mathfrak{b} \subset \mathfrak{p}$ a Borel. By the Levi decomposition, write $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{n}$ and $\mathfrak{b} = \mathfrak{h} \ltimes \tilde{\mathfrak{n}} \subset \mathfrak{p}$. Write $Z(\mathfrak{g}) := Z(U(\mathfrak{g}))$.

Step 1

Our first step is showing that $Z(\mathfrak{g}) \subset Z(\mathfrak{l})$. In fact, we have the following commutative diagramme:

$$\begin{tikzcd}
Z(\mathfrak{g}) \arrow[r, hook] \arrow[dr, hook] & Z(\mathfrak{l}) \arrow[d, hook] & \\
& Z(\mathfrak{h}) \arrow[r, equal] & S(\mathfrak{h})
\end{tikzcd}$$

As an intuition for this result, consider $G = \mathrm{GL}_n$ (and we can extend to any reductive group by considering faithful representations). Fully explicitly, take $G = \mathrm{GL}_5$ and P a subgroup of block-upper-triangular matrices corresponding to the partition $5 = 3 + 1 + 1$, so $P = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$ and $L = \begin{pmatrix} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$. $Z(G) = \lambda I$, $Z(L) = \mathrm{diag}(\lambda_1, \lambda_1, \lambda_1, \lambda_2, \lambda_3)$, and $Z(T) = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$.

Step 2

Clearly $Z(\mathfrak{g}) \subset \bigcap Z(\mathfrak{l})$ where we run over all Levi subgroups. But we can consider only *minimal* Levi subgroups (those corresponding to just one simple root) and this will still be true. So let I be the set of all simple roots. Then we have

$$Z(\mathfrak{g}) \subset \bigcap_{\alpha \in I} Z(\mathfrak{l}_\alpha)$$

Step 3

Once we arrive at a minimal Levi factor, computing its centre is a standard $\mathfrak{sl}_2$-computation, since we now have only one simple root to consider. We can directly compute

$$Z(\mathfrak{l}_\alpha) = Z(\mathfrak{h})^{(s_\alpha, \bullet)}$$

where s_α is the simple reflection corresponding to α and $\bullet$ is the dot-action.

Step 4

We can directly calculate that $Z(\mathfrak{h}) = S(\mathfrak{h})$, so

$$Z(\mathfrak{l}_i) = S(\mathfrak{h})^{(s_i, \bullet)}$$

and

$$Z(\mathfrak{g}) \subset \bigcap_{i \in I} Z(\mathfrak{l}_i) = \bigcap_{i \in I} S(\mathfrak{h})^{(s_i, \bullet)}$$

Any element in this intersection is invariant under all s_i, but s_i generate W, so $Z(\mathfrak{g}) \subset S(\mathfrak{h})^{(W, \bullet)}$.

Step 5

Both the left-hand and right-hand sides are graded $\mathbb{C}$-algebras, so we can conclude by dimension-counting. We can compute $\operatorname{gr} Z(\mathfrak{g}) = \operatorname{gr} S(\mathfrak{h})^{(W,\bullet)}$, so their sizes are correct and we must have $Z(\mathfrak{g}) = S(\mathfrak{h})^{(W,\bullet)}$.

2.3 Feigin-Frenkel

Now, we will discuss a sketch of the proof of the Feigin-Frenkel Centre using the same approach as in the finite-dimensional case. Our first thought might be to construct a direct analogue of the (twisted) Harish-Chandra homomorphism. But this doesn't work! We would get a map $Z(U(\hat{\mathfrak{g}})) \to Z(U(\hat{\mathfrak{l}}))$, which isn't what we want. In fact, $Z(U(\hat{\mathfrak{g}}))$ is not even the right definition of the centre at the critical level!

Definition 4 (vacuum Verma module, [Fre07] 2.2)**.** *Define*

$$V_\kappa(\hat{g}) := U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}\,\kappa)} \mathbb{C}_\kappa$$

then $V_\kappa(\hat{g})$ has the structure of a vertex algebra, called the ***vacuum Verma module*** *for $\mathfrak{g}$.*

Definition 5 (centre of a vertex algebra, [Fre07] 3.3.1)**.** *Let V be a vertex algebra with $Y(-,z)$ its state-field correspondence. Then we define*

$$\mathcal{Z}(V) := \{B \in V | [Y(A,z), Y(B,z)] = 0 \forall A \in V\}$$

When $V = V_\kappa(\hat{g})$, we write $\mathfrak{z}(\mathfrak{g}) := \mathcal{Z}(V_\kappa(\hat{g}))$.

Step 1

The *free field realisation* (which is an entire chapter of Frenkel's book so we will definitely not define it) gives us a commutative diagramme:

$$\begin{tikzcd} \mathfrak{z}(\mathfrak{g}) \arrow[r, hook] \arrow[dr, hook] & \mathfrak{z}(\mathfrak{l}) \arrow[d, hook] & \\ & \mathfrak{z}(\mathfrak{h}) \arrow[r, equal] & S(\mathfrak{h}) \end{tikzcd}$$

Step 2

Then, as before, we write

$$\mathfrak{z}(\mathfrak{g}) \subset \bigcap_{i \in I} \mathfrak{z}(\mathfrak{l}_i)$$

Step 3

Then we do a $\widehat{\mathfrak{sl}_2}$-computation to understand $\mathfrak{z}(\mathfrak{l}_i) \hookrightarrow \mathfrak{z}(\mathfrak{h})$.

Step 4

Then we want to understand $\bigcap_{i\in I} \mathfrak{z}(\mathfrak{l}_i)$.

Step 5

Finally, we show that $\mathfrak{z}(\mathfrak{g})$ is large enough (through the study of Wakimoto modules, this is related to the Segal-Sugawara construction we mentioned but that is not sufficient).

2.4 What Now?

Now that we understand the centre, we are ready to investigate $\mathcal{O}_{\text{crit}}$. We can now do block decomposition, and in particular, some analogue of the Jantzen translation principle allows us to stay in the principal block like before and work on character formulas (and our generalisation of Kazhdan-Lusztig).

3 Characters

3.1 Derivations and the Affine Root

Before we can talk about character formulas, we need to discuss derivations of affine Lie algebras and the affine root.

Definition 6 (derivation). *Let $\mathfrak{g}$ be a Lie algebra. We say a linear map $D : \mathfrak{g} \to \mathfrak{g}$ is a* ***derivation*** *if $D([x, y]) = [D(x), y] + [x, D(y)]$.*

When we consider $\hat{\mathfrak{g}}_\kappa$ our affine Lie algebra, there is a particularly convenient derivation $\delta : \hat{\mathfrak{g}}_\kappa \to \hat{\mathfrak{g}}_\kappa$ given by

$$\delta(a \otimes t^m + b \cdot c) := t\frac{d}{dt}(a \otimes t^m)$$

where $a \in \mathfrak{g}, t^m \in \mathbb{C}((t)), b \in \mathbb{C}$. In particular, $\delta(c) = 0$. We can then define

$$\hat{\mathfrak{g}}_\kappa^{\mathrm{KM}} := \hat{\mathfrak{g}}_\kappa \rtimes \mathbb{C}\, d$$

where $[d, x] = \delta(x)$. **Remark**: this construction does not depend on level at all.

We can also compute

$$\hat{\mathfrak{g}}_\kappa^{\mathrm{KM}} \supset \hat{\mathfrak{h}}^{\mathrm{KM}} = \mathfrak{h} \oplus \mathbb{C}\, c \oplus \mathbb{C}\, d$$

since d commutes with c and $\mathfrak{h}$. Then $\left(\hat{\mathfrak{h}}^{\mathrm{KM}}\right)^* = \mathfrak{h}^* \oplus \mathbb{C}\, \lambda_0 \oplus \mathbb{C}\, \delta$ where we associate λ_0 as the dual of c and by abuse of notation δ as the dual of d.

This element $\delta \in \left(\hat{\mathfrak{h}}^{\mathrm{KM}}\right)^*$ is very important in the study of affine Kac-Moody algebras. The simple roots of $\hat{\mathfrak{g}}_\kappa^{\mathrm{KM}}$ are the simple roots of $\mathfrak{g}$ along with the affine simple root:

Definition 7 (affine simple root, [Bum20]). *Let θ be the highest root of $\mathfrak{g}$. Then $\alpha_0 := \delta - \theta$ is a simple root of $\hat{\mathfrak{g}}_\kappa^{\mathrm{KM}}$ called the* ***affine simple root****.*

In Kac-Moody algebras, we introduce *imaginary* roots, while finite-dimensional Lie algebras have only real roots. We can explicitly describe the roots of $\hat{\mathfrak{g}}_\kappa^{\mathrm{KM}}$ as follows. The real roots are $\{\alpha + n\delta\}$ for α a root of $\mathfrak{g}$ and $n \in \mathbb{Z}_{\geq 0}$. The imaginary roots are $\{n\delta\}$ for $n \neq 0 \in \mathbb{Z}$.

Recall in the finite-dimensional case, we have the *coweight lattice*, which is the lattice $\Lambda^\vee = \{v | \langle \alpha, v\rangle \in \mathbb{Z}\}$ where α is a root.

Definition 8 (affine Weyl group). *Recall that W acts on coweights of $\mathfrak{g}$ by considering them inside $\mathfrak{h}$. Let W^{aff}, associated to $\hat{\mathfrak{g}}_\kappa^{\mathrm{KM}}$, be the group $\Lambda^\vee \rtimes W$, the* ***affine Weyl group****.*

We also recall from the finite-dimensional case $\rho := \frac{1}{2}\sum_{\alpha \in \Delta_+} \alpha$. This is ill-defined here since we have an infinite number of positive roots, but we can define $\hat{\rho} := \rho + h^\vee \lambda_0$. One can check that this $\hat{\rho}$ satisfies the same property as ρ does in the finite-dimensional case: ρ is uniquely determined by $\langle \rho, \alpha_i^\vee \rangle = 1 \forall i$.

In this scenario, we see that W^{aff} and $\hat{\rho}$ are the correct choices:

Conjecture 2 (Feigin-Frenkel,[AF10]). *For λ, μ weights*[4]*, we have $[\overline{\Delta(\lambda)} : L(\mu)] = 0$ unless λ and μ lie in the same $(W^{aff}, \bullet)$-orbit (where this $\bullet$ action is $w \bullet \lambda = w(\lambda + \hat{\rho}) - \hat{\rho}$).*

(I am slightly lying to you by not explaining what $\overline{\Delta(\lambda)}$ is. This work is done in a slightly smaller category than $\mathcal{O}_{\mathrm{crit}}$, and in this category these *reduced Verma modules* are the standard objects. We will not be discussing anything like this, but I wanted to make the point that W^{aff} is important.)

3.2 Characters

This section follows [Ara07].

Introduction

We will be discussing a 2007 paper of Arakawa [Ara07] that extends the Weyl character formula to the critical level:

Theorem 2 (Theorem 1 [Ara07]). *Let $\lambda \in P^+_{crit}$. Then,*

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w \bullet \lambda}}{\prod_{\alpha \in \Delta_+} \left(1 - e^{-\langle \lambda + \rho, \alpha^\vee \rangle \delta}\right) \prod_{\alpha \in \Delta_+^{re}} (1 - e^{-\alpha})}$$

where W is the Weyl group of $\mathfrak{g}$ (not the affine Weyl group), $\ell(w)$ is the length of w, and $w \bullet \lambda$ is the dot-action. Note that Δ_+ denotes the positive roots of $\mathfrak{g}$ and Δ_+^{re} is the positive real roots of $\hat{\mathfrak{g}}_\kappa^{\mathrm{KM}}$.

When $L(\lambda)$ is an integrable representation of $\mathfrak{g}$, the Weyl-Kac formula gives us the character. The Kazhdan-Lusztig Conjecture was extended to the affine case by Kashiwara-Tanisaki ([KT95]) in the negative level and later in the positive level. Combining everything known by 2007, we can find $\operatorname{ch} L(\lambda)$ in any case *other* than the critical level ([Ara07]).

Remark 1. *Adding in our derivation d does not really change the representation theory. We just need to be careful about how d acts when we construct our category $\mathcal{O}_{\mathrm{crit}}$. In particular, all of the results from the previous talk remain true unchanged here.*

Structure of $\mathcal{O}_{\mathrm{crit}}^{\mathrm{KL}}$: Weyl Modules and Linkage

Definition 9 ([Ara07] 2.1). *Let $\mathcal{O}_{\mathrm{crit}}^{\mathrm{KL}}$ be the full subcategory of $\hat{\mathfrak{g}}_\kappa^{\mathrm{KM}}$-mod consisting of objects M such that*

1. *c acts on M by $-h^\vee$ (note that we have already required this in our definition of $\hat{\mathfrak{g}}_\kappa^{\mathrm{KM}}$-mod).*

[4] In the same *critical equivalence class*, but we will ignore this.

2. *d acts on M semisimply: $M = \bigoplus_{a \in \mathbb{C}} M_a$ where $M_a = \{m \in M | d \cdot m = am\}$.*

3. *$\dim M_a < \infty$ for all $a \in \mathbb{C}$.*

4. *There exists a finite set $\{a_1, \ldots, a_n\} \subset \mathbb{C}$ so that $\dim M_a = 0$ unless $a_i - a \in \mathbb{Z}_{\geq 0}$ for some $1 \leq i \leq n$.*

This probably looks quite strange, but this is what I meant by *careful* about how d acts. Any $M \in \mathcal{O}^{\mathrm{KL}}_{\mathrm{crit}}$ admits a weight space decomposition for $\hat{\mathfrak{h}}^{\mathrm{KM}}$: $M = \bigoplus_\lambda M^\lambda$, where $\lambda \in \left(\hat{\mathfrak{h}}^{\mathrm{KM}}\right)^*$ and $M^\lambda = \{m \in M | h \cdot m = \lambda(h) m \forall h \in \hat{\mathfrak{h}}^{\mathrm{KM}}\}$.

Definition 10 (Weyl module, [Ara07] 2.1, [FG07])**.** *The **Weyl module** $V(\lambda)$ is given by*

$$V(\lambda) := U\left(\hat{\mathfrak{g}}^{\mathrm{KM}}_\kappa\right) \otimes_{U(\mathfrak{g} \otimes \mathbb{C}[[t]] \oplus \mathbb{C}\, c)} L_{\mathfrak{g}}(\lambda)$$

where $L_{\mathfrak{g}}(\lambda)$ denotes the irreducible highest-weight representation of $\mathfrak{g}$ of weight λ.

In particular, $V(-h^\vee \lambda_0)$ has the structure of a vertex algebra and is canonically identified with $V\left(\hat{\mathfrak{g}}^{\mathrm{KM}}_\kappa\right)$, the vertex algebra we defined last time.

Remark 2. *Despite some comments I made last time, we will not be using vertex operator algebras at all.*

Theorem 3 ([Ara07] 2.1)**.** *By the Weyl character formula,*

$$\operatorname{ch} V(\lambda) = \frac{\displaystyle\sum_{w \in W} (-1)^{\ell(w)} e^{w \bullet \lambda}}{\displaystyle\prod_{j \geq 1} (1 - e^{-j\delta}) \prod_{\alpha \in \Delta^{\mathrm{re}}_+} (1 - e^{-\alpha})}$$

Given a module $M \in \mathcal{O}^{\mathrm{KL}}_{\mathrm{crit}}$, we can naturally consider M a $V\left(\hat{\mathfrak{g}}^{\mathrm{KM}}_\kappa\right)$-module.

As we discussed last time, being able to find the centre $\mathfrak{z}(\hat{\mathfrak{g}}_\kappa)$ is extremely important to character formulae. We will need to describe the centre to use it, so we refer to the result of Feigin and Frenkel to explicitly describe the centre.

Theorem 4 (Feigin-Frenkel, [Ara07] Theorem 2, [FF92])**.** *There exists $\operatorname{rank}(\mathfrak{g})$ many homogeneous vectors $p^i \in \mathfrak{z}(\hat{\mathfrak{g}}_\kappa)$, $1 \leq i \leq \operatorname{rank}(\mathfrak{g})$, so that there is a $\mathbb{C}$-linear isomorphism*

$$\mathbb{C}[p^i_{(-n)} | 1 \leq i \leq \operatorname{rank}(\mathfrak{g}), n \in \mathbb{Z}_{\geq 1}] \cong \mathfrak{z}(\hat{\mathfrak{g}}_\kappa)$$

where $p^i_{(-n)}$ denotes the $-n$-coefficient of p^i.

In the theory of vertex algebras, we write $p^i_{(-n)}$ in a particular indexing pattern:

$$Y(p^i, z) := \sum_{n\in\mathbb{Z}} p^i_{(n)} z^{-n-1}$$

Now, we shift by the *exponents* of $\mathfrak{g}$: the homogeneous degrees of the generators of $Z(\mathfrak{g})$. Let $d_i, 1 \leq i \leq \operatorname{rank}(\mathfrak{g})$ be the i-th exponent of $\mathfrak{g}$. Then define p^i_n so that

$$Y(p^i, z) = \sum_{n\in\mathbb{Z}} p^i_n z^{-n-d_i-1}$$

and define

$$R_Z := \mathbb{C}[p^i_n | 1 \leq i \leq \operatorname{rank}(\mathfrak{g}), n \in \mathbb{Z}]$$

I don't know what this is called, but every $M \in \mathcal{O}^{\mathrm{KL}}_{\mathrm{crit}}$ can also be treated as a R_Z-module.

Endomorphisms of Weyl Modules

For any dominant weight λ of $\mathfrak{g}$, write $|\lambda\rangle$ for the highest weight vector of $V(\lambda)$.

Definition 11.

$$R^\lambda_Z := R_Z / \operatorname{Ann}_{R_Z} |\lambda\rangle$$

Theorem 5 ([Ara07] Theorem 7)**.** *We have an isomorphism*

$$R^\lambda_Z \cong \operatorname{End}_{U(\hat{\mathfrak{g}}_\kappa)}(V(\lambda))$$

Remark 3. *This is only true when we consider the right-hand side as the endormorphisms of $V(\lambda)$ as $U(\hat{\mathfrak{g}}_\kappa)$-module, not a $U\left(\hat{\mathfrak{g}}^{\mathrm{KM}}_\kappa\right)$-module.*

In particular, since $V(-h^\vee\lambda_0) = V(\hat{\mathfrak{g}}_\kappa)$, by a result of [FF92] we have $\mathfrak{z}(\hat{\mathfrak{g}}) \cong \operatorname{End}_{U(\hat{\mathfrak{g}}_\kappa)} V(\hat{\mathfrak{g}}_\kappa) \cong R^{-h^\vee\lambda_0}_Z$.

Theorem 6 ([Ara07] Theorem 7)**.** *R_Z, and thus R^λ_Z, are graded by d. So set $\operatorname{ch} R^\lambda_Z := \sum_{a\in\mathbb{C}} e^{a\delta} \dim\left(\left(R^\lambda_Z\right)_a\right)$. Then*

$$\operatorname{ch} R^\lambda_Z = \frac{\prod_{\alpha\in\Delta_+}\left(1 - e^{-\langle\lambda+\rho,\alpha^\vee\rangle\delta}\right)}{\prod_{j\geq 1}\left(1 - e^{-j\delta}\right)}$$

Theorem 7.

$$\operatorname{ch}(V(\lambda)) = \operatorname{ch}(L(\lambda)) \cdot \operatorname{ch}(R^\lambda_Z)$$

so

$$
\begin{aligned}
\operatorname{ch}(L(\lambda)) &= \frac{\displaystyle\sum_{w\in W}(-1)^{\ell(w)}e^{w\bullet\lambda}}{\displaystyle\prod_{j\geq 1}(1-e^{-j\delta})\prod_{\alpha\in\Delta_+^{\mathrm{re}}}(1-e^{-\alpha})}\cdot\left(\frac{\displaystyle\prod_{\alpha\in\Delta_+}\left(1-e^{-\left\langle\lambda+\rho,\alpha^\vee\right\rangle\delta}\right)}{\displaystyle\prod_{j\geq 1}\left(1-e^{-j\delta}\right)}\right)^{-1}\\
&= \frac{\displaystyle\sum_{w\in W}(-1)^{\ell(w)}e^{w\bullet\lambda}}{\displaystyle\prod_{\alpha\in\Delta_+}\left(1-e^{-\left\langle\lambda+\rho,\alpha^\vee\right\rangle\delta}\right)\prod_{\alpha\in\Delta_+^{\mathrm{re}}}(1-e^{-\alpha})}
\end{aligned}
$$

And we are done with the proof of the character formula!

4 Opers

5 $\mathcal{W}$-Algebras

We previously saw that the critical level is unique in many ways, in particular that the Virasoro algebra collapses only at the critical level. But we would still love to understand the structure of the Virasoro algebra at the critical level and how it applies to representation theory – so we construct $\mathcal{W}$-algebras, a generalisation of the Virasoro algebra that retains structure at the critical level. We will discuss (classical) $\mathcal{W}$-algebras and their structure, and then introduce affine $\mathcal{W}$-algebras and show that they construct the centre at the critical level.

In particular, we will see that $\mathcal{W}(\mathfrak{sl}_2) = Vir$ (the Virasoro algebra), and by virtue of the existence of $\mathfrak{sl}_2$-triples we have $Vir \subset \mathcal{W}(\mathfrak{g})$. But what happens when Vir doesn't exist (i.e. we are at the critical level)?

We will be able to answer this question next time, when we prove that $\mathcal{W}^{aff}_{G,crit} \cong \operatorname{Fun} \operatorname{Op}_{^LG}(D^\times)$.

Today, we will discuss the construction of the classical (finite) $\mathcal{W}$-algebra. The story is a bit anachronistic because $\mathcal{W}$-algebras were first discovered in the affine case, and later related to the finite-dimensional case.

5.1 Kostant Slices

This section follows [Los].

Principal $\mathfrak{sl}_2$-Triple

Let $\mathfrak{g}$ be a simple Lie algebra over a field k with triangular decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{h} \oplus \mathfrak{u}^+$. Let $\alpha_i, 1 \leq i \leq \operatorname{rank} \mathfrak{g}$ be the simple roots. Let $e_i \in \mathfrak{g}_{\alpha_i}$ be the positive root vector associated to α_i, and f_i the corresponding negative root vector. Let $\mathfrak{b} := \mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{u}^+$.

Write $e = \sum m_i e_i, f = \sum f_i, h = 2\rho^\vee$. Then $[h,e] = 2e, [h,f] = -2f, [e,f] = h$, so (e,f,h) forms a **$\mathfrak{sl}_2$-triple**.

From this information, we can write $\mathfrak{g}$ as a $\mathfrak{sl}_2$-representation via $\operatorname{ad}(e), \operatorname{ad}(h), \operatorname{ad}(f)$. Then let $\mathfrak{g}_i = \{x \in \mathfrak{g} \,|[h,x] = ix\}$. Then

1. $\mathfrak{h} = \mathfrak{g}_0$
2. $\mathfrak{g}_i = \bigoplus\limits_{\alpha} \mathfrak{g}_\alpha$ where we sum over α where $h(\alpha) = i$
3. $\mathfrak{b} = \bigoplus\limits_{i \geq 0} \mathfrak{g}_i, \mathfrak{u}^+ = \bigoplus\limits_{i>0} \mathfrak{g}_i$

Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}$. Then (in the Zariski topology),

1. $\overline{Be} = \mathfrak{u}$
2. $\overline{Ge} = \mathcal{N}$
3. $Gf = Ge$

Slices and Properties

The question then becomes, can we find a transverse slice to Gf in $\mathfrak{g}$? Specifically, this means an affine subspace $S \subset \mathfrak{g}$ with $\forall f \in S, T_f(S) \oplus T_f(Gf) = T_f\,\mathfrak{g} = \mathfrak{g}$. In particular, $T_f(Gf) = [\mathfrak{g}, f] = \operatorname{im}\operatorname{ad}(f)$.

We know that $\ker\operatorname{ad}(e) \oplus \operatorname{im}\operatorname{ad}(f) = \mathfrak{g}$, so $S := f + \ker\operatorname{ad}(e)$. This is the **(principal) Kostant slice**.

Kostant Slice Comparisons

The map $\alpha : U \times S \to f + \mathfrak{b}$ where $(n, s) \mapsto \operatorname{ad}(n)s$ is an isomorphism. Further, $\pi : S \to \mathfrak{g} /\!\!/ G$ is an isomorphism. So in fact $S \cong f + \mathfrak{b}\,/U \cong \mathfrak{g} /\!\!/ G$.

5.2 Finite $\mathcal{W}$-Algebras

This section and the entire next lecture are an extended version of the approach Sam Raskin used to explain this to me. We start with the previous map

$$(f + \mathfrak{b})/U \cong \mathfrak{g} /\!\!/ G$$

by Kostant. If we try to quantise this, first we take the space of functions for each, and then quantise.

$$\begin{tikzcd} f + \mathfrak{b}\,/U \arrow[r, equal] & \mathfrak{g} /\!\!/ G \arrow[d, "\mathrm{Fun}"] \\ & \mathrm{Sym}(\mathfrak{g})^G \arrow[d, "q"] \\ & U(g)^G = Z(G) \end{tikzcd}$$

To quantise $f + \mathfrak{b}\,/U$, we start by consdering $\mathfrak{b}\,/U$. Consider $\mathcal{O}_{\mathfrak{b}} \in \mathrm{QCoh}(\mathfrak{g})$. We can write $\mathfrak{b}^\vee = \mathfrak{g}^\vee / \mathfrak{b}^\perp = \mathfrak{g}^\vee / \mathfrak{u}$, and if we identify $\mathfrak{g} \leftrightarrow \mathfrak{g}^\vee$ we can identify $\mathfrak{b}^\vee = \mathfrak{g} / \mathfrak{u}$. So $\mathcal{O}_{\mathfrak{b}} = U(\mathfrak{g} / \mathfrak{u}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{u})} k = \operatorname{Ind}_{\mathfrak{u}}^{\mathfrak{g}} k$.

Now, consider $f + \mathfrak{b}\,/U$. Let ψ be the character of $\mathfrak{u}$ that takes value 1 on each e_i. In particular, $\psi = k(f, -)$, and $f + \mathfrak{b} = \pi^{-1}(\psi)$, where $\pi : \mathfrak{g} \twoheadrightarrow \mathfrak{g} / \mathfrak{b}$. So somehow the analogy from the result for $\mathcal{O}_{\mathfrak{b}}$ is $\operatorname{Ind}_{\mathfrak{u}}^{\mathfrak{g}} \psi$, and thus the appropriate quantisation of $f + \mathfrak{b}\,/U$ is $(\operatorname{Ind}_{\mathfrak{u}}^{\mathfrak{g}} \psi)^U$. [5]

Definition 12 (classical $\mathcal{W}$-algebra)**.** $\mathcal{W}_G^{fin} := (\operatorname{Ind}_{\mathfrak{u}}^{\mathfrak{g}} \psi)^U$.

1. $\mathcal{W}_G^{fin} \cong \operatorname{End}_{\mathfrak{g}} \operatorname{Ind}_{\mathfrak{u}}^{\mathfrak{g}} \psi$
2. $\mathcal{W}_G^{fin} \cong \mathrm{R}\operatorname{End}_{\mathfrak{g}} \operatorname{Ind}_{\mathfrak{u}}^{\mathfrak{g}} \psi \iff \operatorname{Ext}^i(\operatorname{Ind}_{\mathfrak{u}}^{\mathfrak{g}} \psi, \operatorname{Ind}_{\mathfrak{u}}^{\mathfrak{g}} \psi) = 0$
3. $\operatorname{Hom}_{\mathfrak{g}}(\operatorname{Ind}_{\mathfrak{u}}^{\mathfrak{g}} \psi, M) = (M \otimes -\psi)^{\mathfrak{u}}$

[5] It isn't immediately clear that U acts on $\operatorname{Ind}_{\mathfrak{u}}^{\mathfrak{g}} \psi$ – this is because the action of $\mathfrak{u}$ on $\operatorname{Ind}_{\mathfrak{u}}^{\mathfrak{g}} \psi \otimes (-\psi)$ is locally nilpotent.

This has two immediate corollaries: first, $\forall M \in \mathfrak{g}$-mod, $\mathcal{W}_{\mathfrak{g}}^{fin} \curvearrowright (M \otimes -\psi)^{\mathfrak{u}}$. This motivates another construction, Drinfeld-Sokolov reduction.

Definition 13 (Drinfeld-Sokolov functor)**.** *The **Drinfeld-Sokolov functor**, $\Psi^{fin} : \mathfrak{g}\text{-}mod \to \mathsf{Vect}$ (in the derived sense), given by*

$$\Psi^{fin}(M) := C^{\bullet}(\mathfrak{u}, M \otimes -\psi)$$

where $C^{\bullet}$ represents the Chevalley complex.

Then $\mathcal{W}_G^{fin} := \mathrm{End}(\Psi^{fin})$, and it is a result that as a *dg*-algebra, $\mathcal{W}_G^{fin}$ is concentrated in cohomological degree zero, and is thus a classical algebra.

The second corollary: since M is a $\mathfrak{g}$-module, there is a canonical map $Z(\mathfrak{g}) \to \mathrm{End}_{\mathfrak{g}}(M)$, so in particular, there is a canonical map

$$Z(\mathfrak{g}) \to \mathcal{W}_G^{fin}$$

Theorem 8 (Kostant)**.** *This map is an isomorphism.*

Proof. We sketch the proof. We claim that $\mathcal{W}_G^{fin}$ admits a Kazhdan-Kostant filtration which is bounded from below, and that the canonical map above is filtered (with respect to the PBW filtration on $Z(\mathfrak{g})$). But $\mathrm{gr}^{KK}(\mathcal{W}_G^{fin}) = \mathrm{Fun}(f + \mathfrak{b}/U) = \mathrm{Fun}(\mathfrak{g}/\!\!/G)$ and we are done (this map is filtered and an isomorphism on associated graded). ☐

Theorem 9 (Skryabin)**.** *Let $\mathfrak{g}\text{-}mod^{U,\psi}$ be the full subcategory of $\mathfrak{g}\text{-}mod$ consisting of objects M where $\mathfrak{u}$ acts locally nilpotently on $M \otimes -\psi$. Then*

$$\mathfrak{g}\text{-}mod^{U,\psi} \cong \mathcal{W}_G^{fin}\text{-}mod$$

6 The Centre, II

6.1 An Example

Example 2. *As promised, we will compute* $\mathcal{W}^{fin}(\mathfrak{sl}_2(\mathbb{C}))$ *(by we I mean Vlad, he computed this). We know by Harish-Chandra that* $Z(\mathfrak{sl}_2) \cong \mathbb{C}[\mathfrak{h}]^{S_2} \cong \mathbb{C}[h^2]$*. So we know a posteriori that our answer is just* $\mathbb{C}[h^2]$*.*

Recall that $\mathfrak{sl}_2$ *admits a triangular decomposition*

$$\mathfrak{sl}_2 = \mathfrak{u}^- \oplus \mathfrak{h} \oplus \mathfrak{u}^+$$

where $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$.

We have our $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$*. So we can realise our character* ψ *as* $\psi\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right) = b$ *as a character of* $\mathfrak{u}^+$*. So in this way we can treat* $\mathbb{C}$ *as a* $\mathfrak{u}^+$*-module where* $\mathfrak{u}^+$ *acts by* ψ*.*

$$\begin{aligned} \mathrm{gr}(M) &:= \mathrm{gr}\left(\mathrm{Ind}_{\mathfrak{u}}^{\mathfrak{g}} \psi\right) \\ &:= \mathrm{Sym}(\mathfrak{g}) \otimes_{\mathrm{Sym}(\mathfrak{u})} \psi \\ &= \mathrm{Sym}(\mathfrak{b}^-) && \text{(By PBW)} \\ &= \mathrm{Sym}(\mathfrak{g} \,/\, \mathfrak{u}) \end{aligned}$$

Where $U(\mathfrak{u}^+)$ *acts on* $U(\mathfrak{g})$ *on the right and we write* ψ *to denote* $\mathbb{C}$ *where* $\mathfrak{u}^+$ *acts by* ψ*.*

Then the U*-invariants will be* $Z(U(\mathfrak{g}))$*. (Maybe Vlad can explain this better.)*

6.2 Affine $\mathcal{W}$-Algebras

Semi-Infinite Cohomology

If we try to naïvely upgrade the previous constructions to the affine $\hat{\mathfrak{g}}$, we immediately arrive at a problem: what cohomology makes sense in the affine case? We would love to just write $C^\bullet(\mathfrak{u}((t)), M \otimes -\psi)$, but cohomology of $\mathfrak{u}((t))$ is not defined. But $\mathfrak{u}((t))$ is the colimit of a limit of finite-dimensional Lie algebras, so we would hope to interpolate these finite dimensional Lie algebras by taking a colimit of a limit. Semi-infinite cohomology is a tool that combines cohomology of $\mathfrak{u}[[t]]$ and homology of $\mathfrak{u}((t)) / \mathfrak{u}[[t]]$.

Define $\mathfrak{u}_i^j = \mathrm{ad}_{t^{-i\rho}} \mathfrak{u}[[t]] / t^j \mathfrak{u}[[t]]$, and $\mathfrak{u}_i = \mathrm{ad}_{t^{-i\rho}} \mathfrak{u}[[t]] = \lim_j \mathfrak{u}_i^j$. Then $\mathfrak{u}((t)) = \bigcup_i \mathfrak{u}_i$, and we define

$$H^{\frac{\infty}{2}}(\mathfrak{u}((t)), \mathfrak{u}[[t]], M) := \mathrm{colim}\, C^\bullet(\mathfrak{u}_i; M \otimes \det(\mathfrak{u}_i \,/\, \mathfrak{u}_0))$$

where $C^\bullet$ denotes the Chevalley complex.

Example 3. *When $\mathfrak{u}$ is commutative (e.g. $\mathfrak{g} = \mathfrak{sl}_2$), we have*

$$H^{\frac{\infty}{2}}(\mathfrak{u}((t)), \mathfrak{u}[[t]], M) \cong \left(M^{\mathfrak{u}[[t]]}\right)_{\mathfrak{u}((t))/\,\mathfrak{u}[[t]]}$$

(the coinvariants of the invariants of M).

Quantum Drinfeld-Simpson

Let $\psi : \mathfrak{u}((t)) \to k$ be defined by the composition of maps

$$\mathfrak{u}((t)) \longrightarrow \mathfrak{u}\,/[\mathfrak{u},\mathfrak{u}]((t)) = \bigoplus_\alpha k((t)) \xrightarrow{\Sigma} k((t)) \xrightarrow{Res} k$$

This is the *Whittaker character.*

Definition 14 ([Ras16] 1.10, quantum Drinfeld-Sokolov functor)**.** *Let* ***quantum Drinfeld-Sokolov functor*** *be given by*

$$\Psi(M) = H^{\frac{\infty}{2}}(\mathfrak{u}((t)), \mathfrak{u}[[t]], M \otimes -\psi) : \hat{\mathfrak{g}}_\kappa\text{−mod} \to \mathsf{Vect}$$

where $H^{\frac{\infty}{2}}$ denotes semi-infinite cohomology (replacing Lie algebra cohomology in the classical case).

Note that in the affine case, we actually have two $\mathcal{W}$-algebras: the vertex algebra $\mathcal{W}^{aff}$ and the topological algebra $\mathcal{W}^{as}$. For simplicity we will only define $\mathcal{W}^{aff}$.

Definition 15 (affine $\mathcal{W}$-algebra)**.** $\mathcal{W}^{aff}_{G,\kappa} = \Psi(\mathbb{V}_\kappa)$ *where Ψ is quantum Drinfeld-Simpson reduction.*

Example 4 ([Ara17] 5.22)**.** *Let's discuss $\mathcal{W}^{aff}$ for $\widehat{\mathfrak{sl}}_2$. In general for $\widehat{\mathfrak{sl}}_n$, we have that $\mathcal{W}^{aff}(\widehat{\mathfrak{sl}}_n)$ is generated by $n-1$ fields (as a vertex algebra), so $\mathfrak{sl}_2$ is generated by one field, which is precisely the Virasoro algebra.*

When κ is non-critical, this is just the Virasoro algebra with central charge $1 - \frac{6(\kappa+1)^2}{\kappa+2}$, which is well-defined when $\kappa \neq -2 = -h^\vee$.

Theorem 10 ([Ras16] 1.12.4)**.** *If $\kappa = \kappa_{crit}$, then $\mathcal{W}^{aff}_{G,crit}$ is commutative and equals $\mathbb{V}^{G[[t]]}_{crit}$ (and $\mathcal{W}^{as}$ equals the centre of $U(\hat{\mathfrak{g}}_{crit})$). If κ is non-critical, then $\mathcal{W}^{aff}_{G,\kappa}$ is non-commutative (and in particular contains the Virasoro algebra).*

6.3 Relation to Opers

Theorem 11 ([Ras16] 1.12.1)**.** *We have the following:*

1. *$\mathcal{W}^{aff}_\kappa$ is concentrated in cohomological degree 0, i.e., $\mathcal{W}^{aff}_\kappa = H^0\Psi(\mathbb{V}_\kappa)$*
2. *$\mathcal{W}^{aff}_\kappa$ and $\mathcal{W}^{as}_\kappa$ carry canonical filtrations whose associated graded are (canonical up to a choice of identification $\mathfrak{g} \cong \mathfrak{g}^\vee$) the algebra of functions on the affine Kostant slices $f + \mathfrak{b}[[t]]/U[[t]] \cong (\mathfrak{g} /\!\!/ G)[[t]]$ and $f + \mathfrak{b}((t))/N((t)) \cong (\mathfrak{g} /\!\!/ G)((t))$ respectively*

3. (Feigin-Frenkel Duality)

$$\mathcal{W}^{aff}_{\hat{\mathfrak{g}},\kappa} \cong \mathcal{W}^{aff}_{^L\hat{\mathfrak{g}},^L\kappa}$$

(where $(^L\kappa - \kappa_{crit})^{-1} = \kappa - \kappa_{crit}$).

In particular, if $\kappa = \kappa_{crit}$, then $\mathcal{W}^{aff}_{\hat{\mathfrak{g}},crit} \cong \mathcal{W}^{aff}_{^L\hat{\mathfrak{g}},\infty}$.

Theorem 12.

$$\operatorname{Fun}\operatorname{Op}_{\hat{\mathfrak{g}}}(D) \cong \mathcal{W}^{aff}_{\hat{\mathfrak{g}},\infty}$$

$$\operatorname{Fun}\operatorname{Op}_{\hat{\mathfrak{g}}}(D^\times) \cong \mathcal{W}^{as}_{\hat{\mathfrak{g}},\infty}$$

Proof. We will sketch a proof of the first claim. We will not define what "level ∞" means, but we will say that $\mathbb{V}_\kappa$ becomes $\operatorname{Fun}(\mathfrak{g}[[t]]\,dt)$ as $\kappa \to \infty$. Then, as we discussed, quantum Drinfeld-Sokolov reduction first takes "coinvariants" with respect to $\mathfrak{u}((t))/\,\mathfrak{u}[[t]]$, which just like before transports us to the Kostant slice, so we get $\operatorname{Fun}((f + \mathfrak{b}[[t]])\,dt)$. Then taking invariants with respect to $\mathfrak{u}[[t]]$ gives $\operatorname{Fun}(((f + \mathfrak{b}[[t]])\,dt)/\,\mathfrak{u}[[t]])$, which is precisely opers.

□

Corollary 1.

$$\operatorname{Fun}\operatorname{Op}_{^L\hat{\mathfrak{g}}}(D) \cong \mathcal{W}^{aff}_{^L\hat{\mathfrak{g}},\infty} \cong \mathcal{W}^{aff}_{\hat{\mathfrak{g}},crit} \cong \mathfrak{z}(\hat{\mathfrak{g}})$$

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