Tame and Wild Quivers

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1 Tame and Wild Quivers

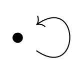
For our talk we fix K an algebraically closed field. There is a vast amount of material to cover today so I will be skipping many details and calculations (and even entire topics) when necessary. If there is time at the end I will touch on some things I skipped.

1.1 Definitions

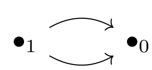
Definition 1 (tame quiver)**.** *We say a connected quiver $\vec{Q}$ is called **tame** if for any $v \in \mathbb{Z}_+^I$, the set of isomorphism classes of indecomposable representations of graded dimension v is the union of a finite number of one-parametre families and a finite number of isolated points.*

This is a significant loosening of the condition of finite type – clearly every finite type quiver is tame but the reverse is not true:

Example 1. *The Jordan quiver is tame but not of finite type. We proved previously that the Jordan quiver is not of finite type, but it is tame: for any dimension d we have precisely one one-parametre family of ireducible representations: J_λ (the Jordan block of size d and eigenvalue λ for any $\lambda \in K$).*

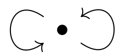


Example 2. *For the Kronecker quiver, we have shown in an earlier talk that when $v = (1, 1)$ our set of indecomposable representations is parametrised by $\mathbb{P}^1$ (which can be thought of as a one-parametre family union a point). We will see later that this quiver is in fact tame.*



Example 3. *For the double loop quiver, we can easily show it is not tame: its path algebra is $K\langle x, y\rangle$, the polynomial algebra in two noncommuting variables. Classifying its irreducible representations is the same as classifying pairs*

of square matrices up to conjugation. This is evidently not tame, as even in the case $d = 1$ we are choosing pairs of scalars $\lambda_1, \lambda_2 \in K$ which is not a one-parametre family.



The double loop quiver is a classical example of a non-tame quiver. In fact, we will define a quiver as *wild* if its irreducible representations cannot be classified without already classifying the representations of the double loop quiver. Formally,

Definition 2 (wild quiver)**.** *Let $\vec{D}$ be the double loop quiver with path algebra $D = K\langle x, y\rangle$. We say an associative algebra A is* ***wild*** *if there is a functor $F : \mathrm{Rep}(D) \to \mathrm{Rep}(A)$ satisfying the following two properties:*

1. *For any $X \in \mathrm{Rep}(D)$, $F(X)$ is indecomposable as a representation of A iff X is indecomposable as a representation of D, and for any $X, Y \in \mathrm{Rep}(D)$, $F(X) \simeq F(Y)$ iff $X \simeq Y$.*
2. *There exists a A, D-bimodule M for which $F(X) = M \otimes_D X$, which is free and finitely generated as a right D-module.*

We say a connected quiver $\vec{Q}$ is wild if its path algebra is wild.

Every fully faithful functor satisfies the first condition, and every functor we use will satisfy the second condition (it is rather technical and we will not get into the details of what it says). As we mentioned before the definition, we are really saying that a quiver is wild if it "contains" the double loop quiver in its representations, for a suitable definition of "contains".

1.2 Tame-Wild Dichotomy

It is clear that a wild quiver cannot be tame, as the prototypical wild quiver contained in every wild quiver is not tame. But it is not clear that every quiver that is not tame is wild.

Theorem 1 (Drozd 1980)**.** *Any algebra that is not tame is wild.*

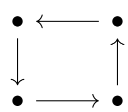
The proof of this fact is outside of the scope of this talk, but it provides a very important dichotomy in the representations of quivers: every quiver is either tame or wild (and not both). This is important because (as we will see later) the representations of tame quivers are easier to work with, and this theorem tells us that every other quiver shares a particular complication. Similar to Gabriel's Theorem that we proved in previous talks, we have a very nice classification of tame quivers in terms of the Tits form. Recall that a quiver is said to be *Dynkin* if the Tits form is positive-definite and *Euclidean* if the Tits form is positive-semidefinite.

Theorem 2. *A connected quiver $\vec{Q}$ is tame if and only if the underlying graph Q is Dynkin or Euclidean.*

In particular a tame but not finite type quiver (such as the Jordan quiver) is Euclidean but not Dynkin. For the rest of the talk, we will introduce a variety of technologies that will come together in a proof of this statement. We will give a high-level sketch of the necessary pieces and show how they come together to prove this theorem, as well as explain why tame quivers are particularly nice in their representations.

2 Affine Root Systems

Now that we have introduced the main concepts we are going to investigate, we will now develop some technology needed in our investigation. Let $\vec{Q}$ be a connected Euclidean quiver. If $\vec{Q}$ has oriented loops, it is a cyclic quiver (example shown below), which can be analysed explicitly. We will not do so for time.



Thus, we can assume $\vec{Q}$ has no oriented cycles (so it is also not the Jordan quiver, which we have proved is tame already anyway). As we have done before, let $L = \mathbb{Z}^I$ be the root lattice of $\vec{Q}$ and $(-,-)$ be the symmetrised Euler form. Let $\{\alpha_i\}_{i\in I}$ be the standard basis of L and W be the Weyl group generated by s_i, the simple reflections. The following construction will be completely black-boxed because developing it fully would take too much time and we will only need it for one result.

There is a Kac-Moody Lie algebra $\mathfrak{g}(Q)$ such that L is the root lattice of $\mathfrak{g}(Q)$ and W is the Weyl group of $\mathfrak{g}(Q)$. For a Euclidean quiver $\vec{Q}$, $\mathfrak{g}(Q)$ is an untwisted affine Lie algebra. If you took Math 216A with Raphael in the Fall, this is the same construction we used, starting from the generalised Cartan matrix generated from $\vec{Q}$. This construction will be completely black-boxed for the purposes of this talk, but the result we obtain from it will not need any of the details involved.

Lemma 1. *Let R denote the roots of $\mathfrak{g}(Q)$, and R^{re}, R^{im} denote the real, imaginary (respectively) roots. Then $R^{re} = \{\alpha \in L \setminus \{0\} | (\alpha, \alpha) = 2\}$, $R^{im} = \{\alpha \in L \setminus \{0\} | (\alpha, \alpha) = 0\}$.*

We can do better than this and explicitly describe our real and imaginary roots. Fix some i_0 so that $Q_f := Q \setminus \{i_0\}$ is a connected graph. We call this i_0 an *extending vertex*. Then denote $L_f = \mathbb{Z}^{I\setminus\{i_0\}}$, and denote the corresponding root system R_f and Weyl group W_f. We note that Q_f is Dynkin, so R_f, W_f are finite. Recall that there is some $\delta \in R_+$ that generates the radical of $(-,-)$.

Theorem 3 (Kac 1990)**.** *For any extending vertex i_0,*

1. $R^{re} = \{\alpha + n\delta | \alpha \in R_f, n \in \mathbb{Z}\}$ *and* $R^{im} = \{n\delta | n \in \mathbb{Z}, n \neq 0\}$.
2. *Given* $\alpha \in L_f$, *we can define* $\tau_\alpha(x) := x + (\alpha, x)\delta$ *which is an automorphism* $L \to L$. *Then set* $T = \{\tau_\alpha | \alpha \in L_f\} \cong L_f$ *so that* $W \cong W_f \ltimes T$.
3. $\forall w \in W$, $w(\delta) = \delta$.

2.1 Affine Coxeter Element

Let $r = |I| - 1$, the number of vertices of $\vec{Q}_f = \vec{Q} \setminus \{i_0\}$. Let C be the Coxeter element of W. Unlike the Dynkin case, C may not have finite order and may be less well-behaved. But we do have the following result:

Theorem 4. *We have*

1. *some* $g > 0$ *so that* $C^g \in T$.
2. $C\alpha = \alpha$ *iff* $\alpha = n\delta$, *for roots* α.

Definition 3. *Let* g, λ *be so that* $C^g = \tau_\lambda$. *For any* $\alpha \in L$, *we define its* ***defect*** $\partial(\alpha) := (\alpha, \lambda) \in \mathbb{Z}$, *so that* $C^g(\alpha) = \alpha + \partial(\alpha)\delta$.

When V is a representation of $\vec{Q}$, we say $\partial(V) := \partial(\dim V)$.

Theorem 5. *Let* $\alpha \in R_+$. *Then*

1. $\partial(\alpha) < 0 \iff C^N\alpha \in R_- \forall N \gg 0$
2. $\partial(\alpha) > 0 \iff C^{-N}\alpha \in R_- \forall N \gg 0$
3. $\partial(\alpha) = 0 \iff C^g\alpha = \alpha$

For any $k = 0, \dots, r$, define $p_k = s_{i_0} \dots s_{i_k}(\alpha_{i_k})$ and $q_k = s_{i_r} \dots s_{i_{k+1}}(\alpha_{i_k})$. Then $Cp_k = -q_k$. We will not need this, but these roots are relevant because $\partial(\alpha) < 0 \iff C^N\alpha \in R_- \forall N \gg 0 \iff \alpha = C^{-n}p_k$ for some $n \geq 0$, and similarly $\partial(\alpha) > 0 \iff C^{-N}\alpha \in R_- \forall N \gg 0 \iff \alpha = C^n q_k$ for some $n \geq 0$.

Example 4.

Example 7.15. Consider the Kronecker quiver:

Then the corresponding root system is given by

$$R^{re} = \{\pm\alpha_1 + n\delta,\ n \in \mathbb{Z}\},$$
$$R^{im} = \{n\delta, n \neq 0\},$$

where $\delta = \alpha_0 + \alpha_1$ (see Figure 7.2). This is exactly the root system of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$.

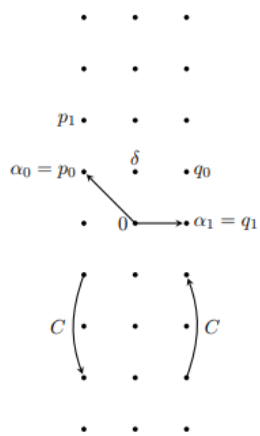


Figure 7.2. Root system $\widehat{A}_1$ and the action of the affine Coxeter element C.

The Coxeter element is $C = s_1 s_0$ (it is chosen to be adapted to the quiver $\vec{Q}$). It acts on the root lattice by

$$C(\alpha_1) = \alpha_1 + 2\delta, \qquad C(\alpha_0) = \alpha_0 - 2\delta, \qquad C(\delta) = \delta.$$

Comparing it with the definition of defect, we see that in this case $g = 1$, $\partial(\alpha_1) = 2$, $\partial(\alpha_0) = -2$. The roots p_i, q_i defined by (7.2) are

$$p_0 = \alpha_0, \qquad p_1 = s_0(\alpha_1) = \alpha_1 + 2\alpha_0,$$
$$q_0 = s_1(\alpha_0) = \alpha_0 + 2\alpha_1, \qquad q_1 = \alpha_1.$$

2.2 Preprojective, Preinjective, and Regular Representations

Recall that we have introduced the Coxeter endofunctors $\mathbf{C}^{\pm}$ on $\text{Rep}(\vec{Q})$, and defined certain classes of representations in terms of them. Given an indecomposable representation V of $\vec{Q}$, we say V is

1. *preprojective*, if $(\mathbf{C}^{+})^n V = 0$ for $n \gg 0$.
2. *preinjecive*, if $(\mathbf{C}^{-})^n V = 0$ for $n \gg 0$.
3. *regular*, $(\mathbf{C}^{-})^n V \neq 0, (\mathbf{C}^{+})^n V \neq 0 \forall n \in \mathbb{Z}_{>0}$.

We have a similar classification for positive roots: we say $\alpha \in R_+$ is

1. *preprojective*, if $C^n \alpha \not> 0$ for some $n > 0$.
2. *preinjecive*, if if $C^n \alpha \not> 0$ for some $n < 0$.
3. *regular*, if $C^n \alpha > 0$ for any $n \in \mathbb{Z}$.

We know that if α is a positive root, then it is preprojective iff $\partial(\alpha) < 0$, preinjective iff $\partial(\alpha) > 0$, and regular iff $\partial(\alpha) = 0$. Then, using the bijection between preprojective roots and preprojective representations that Sam mentioned last time (and similar for preinjective), we arrive at the following result.

Theorem 6. *Let I be an indecomposable representation of $\vec{Q}$. Then I is*

1. *preprojective if $\partial(I) < 0$*
2. *preinjective if $\partial(I) > 0$*
3. *regular if $\partial(I) = 0$*

Further, for any positive root α with $\partial(\alpha) \neq 0$, we have a unique (up to isomorphism) indecomposable representation I of $\vec{Q}$ with $\dim(I) = \alpha$, with I preprojective if $\partial(\alpha) < 0$ and I preinjective if $\partial(\alpha) > 0$.

In particular, we have a bijection

$$\{\text{preprojective and preinjective representations of } \vec{Q}\}/\simeq$$

$$\cong$$

$$\{\text{positive roots } \alpha \in R | \partial(\alpha) \neq 0\}$$

so we have a complete classification of preinjective and preprojective representations. The last step is to classify regular representations. The real understanding behind our classification of regular representations comes from McKay correspondence and subgroups of SU(2), but unfortunately there are not enough weeks in the quarter to devote a talk to that topic. So we will give a sketch of the ideas behind the classification.

We have a full subcategory $\mathcal{R}(\vec{Q})$ (which is an abelian category closed under extensions) of $\mathrm{Rep}(\vec{Q})$ consisting of regular representations of $\vec{Q}$, which is equipped with endofunctors $\mathbf{C}^+$ and $\mathbf{C}^-$, which are autoequivalences by definition of regular representations. Thus, it makes sense to consider the $\mathbf{C}^+$-orbit of a simple regular representation. Fix X a simple regular representation of $\vec{Q}$, and $\mathcal{O} := \{X, \mathbf{C}^+X, \ldots, (\mathbf{C}^+)^{l-1}(X)\}$, where l is called the *period* of X – the minimal positive integer so that $(\mathbf{C}^+)^l X = X$. Such an integer always exists. Given an orbit $\mathcal{O}$ of X, we define $\mathcal{R}_{\mathcal{O}}$ as the full subcategory of $\mathcal{R}(\vec{Q})$ given by $V \in \mathcal{R}(\vec{Q})$ where the composition series of V contains only elements of $\mathcal{O}$. We define the *tube* of $\mathcal{O}$ as the set of all indecomposables in $\mathcal{R}_{\mathcal{O}}$.

Theorem 7. *We have*

1. *a bijection $\mathbb{P}^1 \cong \{$tubes of $\vec{Q}\}$.*
2. *the set of tubes of period greater than* 1 *is finite.*

3 Conclusion

3.1 Euclidean Quivers are Tame

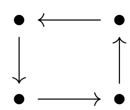
Theorem 8. *Let $\vec{Q}$ be a connected Euclidean quiver that is not the Jordan quiver, and R its root system.*

1. *An indecomposable representation of $\vec{Q}$ of dimension $v \in L$ exists if and only if $v \in R_+$.*
2. *For any real positive root α, the indecomposable representations of $\vec{Q}$ of dimension α is unique up to isomorphism*
3. *There is a finite subset $D = \{p_1, \ldots, p_k\} \subset \mathbb{P}^1$ and a collection of integers $l_p > 1, p \in D$ so that for any positive imaginary root $\alpha = n\delta$, the set of isomorphism classes of indecomposable representations of $\vec{Q}$ with dimension α is in bijection with the set*

$$(\mathbb{P}^1 \setminus D) \cup \bigcup_{p \in D} \mathbb{Z}_{l_p}$$

Proof. First assume $\vec{Q}$ is such that the extending vertex is a sink and let $\alpha \in R_+$ a positive real root. We will first show there is exactly one indecomposable representation of $\vec{Q}$ with dimension α. If $\partial(\alpha) > 0$, then there is a unique preinjective representation (up to isomorphism) of dimension α. If $\partial(\alpha) < 0$, then there is a unique preprojective representation (up to isomorphism) of dimension α. Now, if α is an arbitrary positive root that is not real, we must have $\partial(\alpha) = 0$. So we are dealing with regular representations. We know the set of greater-than-period-1 tubes is finite, so denote its image under the bijection in $\mathbb{P}^1$ by D. Then we have a bijection between regular representations and $(\mathbb{P}^1 \setminus D) \cup \bigcup_{p \in D} \mathbb{Z}_{l_p}$.

Then, if i_0 is not necessarily a sink, we note the other cases are that either $\vec{Q}$ is a cyclic quiver (example below) or i_0 is a source. The proof for cyclic quivers is very straightforward but we will omit it for time. If i_0 is a source, we simply apply the reflection functor and then use the previous argument.



□

Corollary 1. *All Euclidean quivers are tame.*

When combined with the fact we are about to prove (that all non-Euclidean quivers are wild) this theorem also provides a complete classification of the indecomposable representations of tame quivers.

3.2 Non-Euclidean Quivers are Wild

We will actually prove a slightly stronger statement: non-Euclidean quivers are *strongly wild*, which, as the name implies, is (strictly) stronger than wild. We will slightly depart from the book in an attempt to spend less time on technical details.

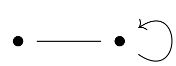
Definition 4. *Let $\vec{Q}$ and $\vec{Q}'$ be quivers. We say $\mathrm{Rep}(\vec{Q}) \subset \mathrm{Rep}(\vec{Q}')$ ("$\mathrm{Rep}(\vec{Q})$ **embeds in** $\mathrm{Rep}(\vec{Q}')$") if there is a functor $T : \mathrm{Rep}(\vec{Q}) \to \mathrm{Rep}(\vec{Q}')$ such that:*

1. *T is fully faithful.*
2. *There exists a $k\vec{Q}', k\vec{Q}$-bimodule M for which $T(V) = M \otimes_{k\vec{Q}} V$, which is free and finitely generated as a right $k\vec{Q}$-module.*

Definition 5. *Recall that we denote $\vec{D}$ as the double loop quiver. We say a quiver $\vec{Q}$ is **strongly wild** if $\mathrm{Rep}(\vec{D}) \subset \mathrm{Rep}(\vec{Q})$.*

Given our definition of strongly wild, it is immediate that every strongly wild quiver is wild. Let's quickly look at some results we will need to prove that every non-Euclidean quiver is strongly wild (and hence wild).

Lemma 2. *The tadpole quiver $\vec{T}$ is strongly wild (for any orientation).*



Lemma 3. *Let $\vec{Q}^E$ be a Euclidean quiver. Then* $\mathrm{Rep}(\vec{J}) \subset \mathrm{Rep}(\vec{Q}^E)$. *where $\vec{J}$ is the Jordan quiver.*

Lemma 4. *Every connected nontrivial non-Euclidean quiver $\vec{Q}$ contains a nontrivial Euclidean quiver $\vec{Q}^E$ as a proper subquiver.*

Now, finally, we can finish our proof that a quiver is tame if and only if it is Euclidean.

Theorem 9. *If $\vec{Q}$ is a connected quiver that is not Euclidean or Dynkin, then $\vec{Q}$ is strongly wild and hence wild.*

Proof. Consider $\vec{Q}^E$, the nontrivial Euclidean quiver that is a proper subquiver of $\vec{Q}$. Since $\vec{Q}^E$ is a *proper* subquiver, there is either a vertex $i \in \vec{Q}$ that is not in $\vec{Q}^E$ or an edge $h \in \vec{Q}$ that is not in $\vec{Q}^E$. We split into cases:

1. There is a vertex $i \in \vec{Q}$ that is not in $\vec{Q}^E$. Then consider the embedding $\mathrm{Rep}(\vec{J}) \subset \mathrm{Rep}(\vec{Q}^E)$. Since $\vec{Q}$ is connected, there must be at least one edge involving i that starts or ends in $\vec{Q}^E$. Thus, we can extend our representation of $\vec{J}$ to a representation of the tadpole quiver by considering this edge involving i, which shows $\mathrm{Rep}(\vec{T}) \subset \mathrm{Rep}(\vec{Q})$. Since $\vec{T}$ is strongly wild, we see $\vec{Q}$ is as well.

2. There is an edge $h \in \vec{Q}$ that is not in $\vec{Q}^E$, but all vertices of $\vec{Q}$ are the same as $\vec{Q}^E$. Then similarly, our embedding $\mathrm{Rep}(\vec{J}) \subset \mathrm{Rep}(\vec{Q}^E)$ extends to an embedding $\mathrm{Rep}(\vec{D}) \subset \mathrm{Rep}(\vec{Q})$, so by definition $\vec{Q}$ is strongly wild.

□

Corollary 2. *Let $\vec{Q}$ be a connected quiver.*

1. *If $\vec{Q}$ is Dynkin, it is of finite type and tame.*

2. *If $\vec{Q}$ is Euclidean and not Dynkin, it is tame and not of finite type.*

3. *If $\vec{Q}$ is not Euclidean or Dynkin, it is wild.*