Stable Homotopy: What Are Stable Phenomena?

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Outline

What is a homotopy?

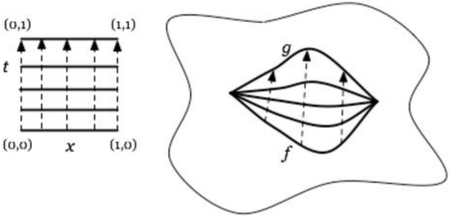


Figure: A homotopy (image from nlab)

Homotopies are a formal way of expressing if two functions are "similar".

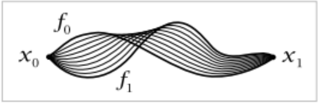
Formal Definition

If you've seen the definition of a homotopy before, this is a brief refresher. If not, feel free to ignore it.

Definition (homotopic)

Let X be a topological space and $x_0, x_1 \in X$. Let f_0 and f_1 be two continuous functions from $[0,1] \to X$ so that $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1) = x_1$. We call f_0 and f_1 **homotopic** if there is a family of functions $f_t : [0,1] \to X$ for $0 \leq t \leq 1$ with the following conditions:

1. f_t is continuous for any $0 \leq t \leq 1$
2. $f_t(0) = x_0, f_t(1) = x_1$ for any $0 \leq t \leq 1$
3. the function $F(s,t) : [0,1] \times [0,1] \to X$ defined by $F(s,t) = f_t(s)$ is continuous



Homotopic Spaces

We can extend the idea of homotopies to spaces themselves to get a sense of when two spaces are "similar". Once again, if you are unfamiliar with the formal terminology, don't worry – it's not important, it's just there if you are curious.

Definition (homotopic II)

Let X and Y be two topological spaces and let f, g be continuous mappings with $f : X \to Y$ and $g : Y \to X$. If $f \circ g$ is homotopic to the identity in Y and $g \circ f$ is homotopic to the identity in X, we call X and Y homotopic.

Outline

Fundamental Groups

The goal of algebraic topology is to classify and differentiate topological spaces using algebraic ideas. One extremely useful tool we have is the "fundamendal group". The **fundamental group** of a space X is the group of all loops, *up to homotopy*. What does that mean?

Fundamental Groups Explained

A loop is called **contractible** if it is homotopic to a point. The term **up to homotopy** means we treat two loops as the same if they are homotopic, so all contractible points are treated as the identity in the fundamental group.

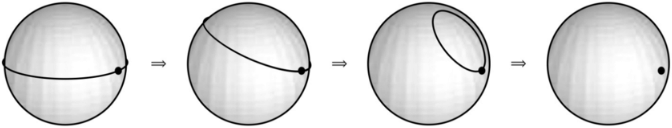


Figure: A contractible loop

A noncontractible loop

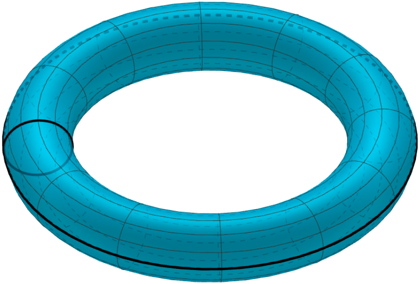


Figure: Some noncontractible loops

Detecting Holes in our Space

You may notice that every loop on a space X is a continuous map from S^1 (the unit circle) into the space. When a loop is not contractible, we see there must be a 1-dimensional hole in the space within that loop.

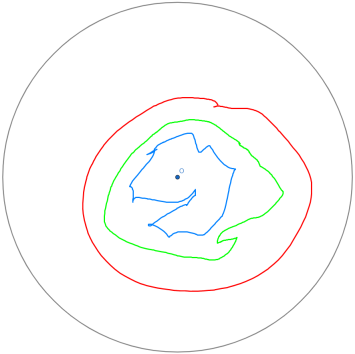


Figure: Some noncontractible loops

Higher Homotopy Groups

What if we want to detect higher dimensional holes? We can instead consider “higher-dimensional loops:” maps from S^k (the k-dimensional sphere) into our space. If X is a topological space, we write $\pi_k(X)$ for the **k-th homotopy group of X**. This group is composed of every k-dimensional loop (map from S^k into our space) up to homotopy.

Outline

This is where the pretty pictures end – I can't draw it anymore. The solution to our dilemma is stable homotopy groups. We need some way to combine all of the data in the homotopy groups, and there are an infinte number of them, so this might be a challenge. But we have a tool in category theory called a **colimit** that lets us do this.

What's Better About This?

1. Better algebraic properties
2. Easier to calculate
3. Easier to generalise

Conclusion

Thank you!