Split Semisimple Groups

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1 Split Semisimple and Split Reductive Groups

Throughout, we fix some arbitrary field F. Our goal today will to be continue with what Jas told us about last week, by introducing split semisimple groups, some of the motivations and results behind them, and ultimately classifying them. The end goal is to demonstrate a method of classifying split simple groups and explain the motivation behind this setup. We will also see how this method generalises to split reductive groups.

1.1 Definitions and Facts

A brief notational disclaimer: in the book T^* is used in root systems to denote the dual of T, ie. $\mathrm{Hom}(T, \mathbb{G}_m)$, while in the seminar we have used $X^\bullet(T)$ to denote the same thing. I will stick to the notation of $X^\bullet(T)$ in root systems to align with previous talks, although in some contexts (that are not root systems) the upper star dual notation is used for clarity.

Definition 1. *An algebraic group G is **solvable** if $G(F_{alg})$ is solvable.*

Definition 2. *An algebraic group G is **semisimple** if G is connected, and $G_{F_{alg}}$ has no nontrivial solvable connected normal subgroups. (Note: in the book, we require $G \neq 1$, which is certainly not simple, but it is in fact semisimple. This can be seen later when we define semisimplicity in terms of root systems, as $G = 1$ has an empty root system which is the sum of precisely 0 irreducible root systems).*

This definition of semisimplicity does in fact reflect the classical case which we will see soon.

Definition 3. *(From now on, "group" means algebraic group). A semisimple group G called **split** if it contains a split maximal torus $T \subset G$.*

Today we will be focusing on the theory and classification of split semisimple groups. The splitness of the torus allows us to investigate the structure of T and G via the theory of diagonalisable groups. Take the adjoint representation of G on $\mathbf{GL}(\mathrm{Lie}(G))$, and restrict it to T. We get

$$\mathrm{Lie}(G) = \bigoplus_{\alpha \in X^\bullet(T)} V_\alpha.$$

The nonzero weights are called **roots** of G (with respect to T).

Definition 4. *Given a root system Φ on V, we have two associated lattices, the **root lattice** and the **weight lattice**: Λ_r is called the **root lattice** and is the lattice in V generated by Φ, and Λ is the **weight lattice**:*

$$\Lambda := \{v \in V | \langle \alpha^*, v\rangle \in \mathbb{Z} \forall \alpha \in \Phi\}.$$

(Λ is dual to the lattice generated by Φ^* in V^*).

These lattices will be immensely useful in our classification of split semisimple groups.

Proposition 1. *Letting $\Phi(G)$ denote the set of roots of G, we have $\Phi(G)$ is a root system in $T^* \otimes_{\mathbb{Z}} \mathbb{R}$. Furthermore, $\Lambda_r \subset T^* \subset \Lambda$ (with respect to $\Phi(G)$).*

1.2 Central Isogenies

Definition 5. *Let G and H be group schemes and $f : G \to H$ be a surjective morphism (this has been defined previously). We say f is a **isogeny** if $\ker(f)$ is finite. We further say f is **central** if $(\ker(f))(R)$ is central in $G(R)$ for any $R \in \mathsf{Alg}_F$.*

Isogenies are the "isomorphisms" of group schemes: in fact, before Weil introcuced the terminology of isogenies, such morphisms were in fact called isomorphisms despite $\ker(f)$ not being necessarily trivial.

I want to detail a couple very nice properties about (central) isogenies. Let G and G' be semisimple groups with G' split, and $f : G \to G'$ be a central isogeny. Then select a maximal split torus T' in G'. Define $T := f^{-1}(T')$. Then T is a maximal split torus in G (so G is split) and the natural map $T'^* \to T^*$ induces an isomorphism $\Phi(G') \xrightarrow{\sim} \Phi(G)$.

Let's look a very important property of isogenies that ties in to the special types of split groups we mentioned earlier. Let G be a spit semisimple group and T its split maximal torus. Then the adjoint representation of G has a kernel N that is a subgroup of T. Because T is split, N is diagonalisable. The restriction of $T^* \to N^*$ has kernel Λ_r, so $N^* \simeq T^*/\Lambda_r$. Further, $\overline{G} := G/N$ is an adjoint group (it is the image of the adjoint representation). Finally, when G is simply connected, we have $T^* = \Lambda$ by definition, so $N^* \simeq \Lambda/\Lambda_r$. The important result now arrives:

Proposition 2. *If G' is any split semisimple group with $\Phi(G') \simeq \Phi(G)$, then $G' \simeq G/K$ where K is an arbitrary subgroup of N.*

We get a correspondence theorem of sorts between split semisimple groups with isomorphic root systems and certain factor groups of G. This result has a little bit of additional structure in the language of Cartier duals, but we have not developed this technology so we will not investigate this further. We also have an immediate consequence of this proposition:

Proposition 3. *For any split semisimple group G, there is a simply connected group $\tilde{G}$ and an adjoint group $\overline{G}$ so that*

$$\tilde{G} \xrightarrow{\tilde{f}} G \xrightarrow{\overline{f}} \overline{G}.$$

Where $\tilde{f}, \overline{f}$ are central isogenies. Moreover, $\tilde{f}$ and $\overline{f}$ are unique up to automorphisms: if $\tilde{f}, \tilde{f}'$ are two such central isogenies, then there is $\alpha \in \mathrm{Aut}(\tilde{G})$ so that $\tilde{f}' = \tilde{f} \circ \alpha$, and similar for $\overline{f}$.

In a sense, we see that for any split semisimple group G, there is a collection of groups with isomorphic root systems, with the "largest" one being simply connected and the "smallest" one being adjoint. Effectively, a split semisimple group G is a split torus combined with some combinatorial data to distinguish between some lattices. This can be formalised a little bit with the following few ideas:

Definition 6. *Let G and G' be split semisimple groups. We say G and G' are a **isomorphic** if there is an isomorphism $f : \Lambda(G) \to \Lambda(G')$ identifying $\Lambda_r(G)$ and $\Lambda_r(G')$, and T^* and T'^*.*

Proposition 4. *Given any root system Φ and additive group $\Lambda_r \subset A \subset \Lambda$, there is a split semisimple group G such that $(\Phi(G), T^*) \cong (\Phi, A)$. Furthermore, there are only finitely many such A (ie. Λ/Λ_r is finite).*

As a result, a split semisimple group is uniquely determined by the lattices Λ, Λ_r and the position of T^* between them.

Corollary 1. *Every split semisimple group is isomorphic to a factor group of a simply connected group.*

Hopefully some of the motivation for these definitions and approach is clear. We are about to see how useful these concepts are in the classification of split semisimple groups.

2 Simple Split Groups

Now that we've had a chance to look at some of the motivations and properties of split semisimple groups, let's investigate one of the most important results. In algebra, there are numerous contexts where it is natural to consider semisimple objects, and a natural question is always: *Can we classify **simple** objects?* And in this context, we have a slightly more specific question: *Can we classify simple objects **using tori and combinatorial data**?* Let's see!

Definition 7. *A split semisimple group G is **simple** if $\Phi(G)$ is irreducible.*

Recall that previously we defined simply connected and adjoint split semisimple groups. In these specific cases, we have another stronger condition:

Proposition 5. *Let G be a split semisimple group that is simply connected or adjoint. Then there exists split semisimple groups G_i that are simple and satisfy*

$$G = \bigoplus_i G_i, \Phi(G) = \sum_i \Phi(G_i).$$

Further, these groups are determined uniquely up to ordering.

In particular, simply connected split semisimple groups are uniquely determined by their root systems (and the same for adjoint). And since every split semisimple group lies between a simply connected one and an adjoint one, it suffices to classify the simply connected and adjoint split simple groups. (In fact it suffices to just classify simply connected simple groups since any split semisimple group is a factor group of a simply connected one, which decomposes as a sum of simply connected simple groups.)

2.1 Reductive Groups

Before we go on to classifying split simple groups, we should develop our theory for reductive groups in the hope of also classifying reductive groups with the same methods. Forget the definition of reductive groups from last time as we will use a completely different (but equivalent) definition here (proof of equivalence is left as an exercise):

Definition 8. *We say an algebraic group G is* ***reductive*** *if*

1. *G is smooth and connected*
2. *$G' = [G, G]$ is semisimple*
3. *$Z(G)$ is diagonalisable*
4. *$f : Z(G) \times G' \to G$ is a central isogeny*

In fact, we have $f' : T \times G' \to G$ is a central isogeny that commutes with $T \hookrightarrow Z(G)$. Let G be a split reductive group. Recall that we can classify split reductive groups via a root datum, which is data of the form $(R, X^\bullet(T), R^\vee, X_\bullet(T))$. Denote $T' \subset G'$ as the split maximal torus in G'. But $R = \Phi(G')$ and we have the following commutative diagramme:

$$\begin{array}{ccccc} & & T^* & & \\ & \nearrow & \downarrow & \searrow & \\ \Lambda_r & \subset & T'^* & \subset & \Lambda \end{array}$$

Ultimately we get $\Lambda_r \hookrightarrow T^* \to \Lambda$, although the second map may not be injective in this case. So our classification is almost identical: we still classify our groups by a diagramme $\Lambda_r \to T^* \to \Lambda$.

2.2 Classifications

For the remainder of the talk (unless otherwise specified), “groups” are split semisimple groups, “simple groups” are split semisimple groups that are simple, “simply connected groups” are simple groups that are simply connected, and “adjoint groups” are simple groups that are adjoint. Now that we know what our irreducible objects look like in this context, we proceed to wonder if we can classify them. The answer is that we can indeed classify simple groups by Lie family. We will only discuss types A, B, C, D today. Our general approach for each case will be as follows:

1. Construct an archetypal example (in the vein of $\mathfrak{sl}_2$ for type A)
2. Calculate the root system of this example
3. Classify the entire Lie family via our root system

As we discussed earlier, split semisimple groups are effectively tori combined with some combinatorial data, so at this point we wish to describe the combinatorial data that can be associated to certain types of split semisimple groups.

2.2.1 Type A

Fix some $n > 0$. We begin with Step 1 and create an archetypal group of type A_n. As you may expect, we take some F-vector space V of dimension $n+1$ and set $G := \mathbf{SL}(V)$. Because V is finite dimensional, we can embed G into $\mathbf{GL}_{n+1}(F)$. This allows us to move on to Step 2: calculating our root system. The first and most important step is finding a maximal split torus: we can embed $F^{n+1} \hookrightarrow \mathbf{GL}_{n+1}(F)$ via the diagonal embedding. Then, if we require $f_{n+1} = \frac{1}{\prod f_i}$, then we get an embedding $F^n \hookrightarrow G$, again via the diagonal embedding. If we denote the image of this embedding as T, we see T is our maximal split torus. Then, we want to try to understand what the character group $X^\bullet(T)$[1] looks like. Set

$$\chi_i(\mathrm{diag}(t_1, \ldots, t_{n+1})) = t_i.$$

Then $X^\bullet(T)$ looks like

$$\mathbb{Z}^n/(e_1 + e_2 + \cdots + e_{n+1})\mathbb{Z}$$

Our next step is to find the weight spaces of the adjoint action of T on $\mathrm{Lie}(G)$. Recall that $\mathrm{Lie}(G)$ is traceless matrices. The weight subspaces end up being

1. T itself, with the trivial weight
2. $F \cdot E_{ij}$ for $1 \leq i \neq j \leq n+1$, with weight $\chi_i \cdot \chi_j^{-1}$

[1]Note: in the book, we use T^* (T dual) to denote this

Thus our root system is $\{\overline{e_i} - \overline{e_j} | i \neq j\}$. Now, remember from earlier that $\Lambda_r \subset T^* \subset \Lambda$, and in this case, $T^* = \sum \mathbb{Z} \cdot \overline{e_i} = \Lambda$, so G is simply connected. The kernel of the adjoint representation of G is its centre, μ_{n+1}. Overall, we arrive at the following classification:

Theorem 1 (Classification of A_n)**.** *A simply connected group G of type A_n is isomorphic to $\mathbf{SL}(V)$ for V a $n+1$-dimensional F-vector space. All other groups are isomorphic to $\mathbf{SL}(V)/\mu_k$ where k divides $n+1$.*

Remark: $\mathbf{PGL}(V) \simeq \mathbf{SL}(V)/\mu_{n+1}$ is adjoint.

2.2.2 Type B

We will proceed with our classification of simple groups, albeit with fewer details for the remaining Lie families. Step 1: Let (V, q) be an F-vector space of dimension $2n+1$ with a regular quadratic form q. We consider $G = \mathbf{O}^+(V, q)$ embedded into $\mathbf{GL}_{2n+1}(F)$. Our maximal torus T is the group of all $t = \text{diag}(1, t_1, \ldots, t_n, t_1^{-1} \ldots, t_n^{-1})$. We define $\chi_i(t) = t_i$ and naturally associate $X^\bullet(T)$ with $\mathbb{Z}^n$.

Step 2: We start by calculating $\text{Lie}(G) \subset \text{End}(V)$, which must satisfy $b_q(v, xv) = 0$ (b_q is the polar form) for any $v \in V$ and $\text{tr}(x) = 0$ for $x \in \text{Lie}(G)$. Then the weight subspaces of $\text{Lie}(G)$ with respect to the adjoint action of T are

1. Diagonal matrices with trivial weight
2. $F \cdot (E_{i,j+n} - E_{j,i+n})$ for $1 \leq i < j \leq n$ with weight $\chi_i \cdot \chi_j$
3. $F \cdot (E_{i+n,j} - E_{j+n,i})$ for $1 \leq i < j \leq n$ with weight $\chi_i^{-1} \cdot \chi_j^{-1}$
4. $F \cdot (E_{i,j} - E_{j+n,i+n})$ for $1 \leq i \neq j \leq n$ with weight $\chi_i \cdot \chi_j^{-1}$
5. $F \cdot (E_{0,i} - 2q(v_0)E_{i+n,0})$ for $1 \leq i \leq n$ with weight χ_i^{-1}
6. $F \cdot (E_{0,i+n} - 2q(v_0)E_{i,0})$ for $1 \leq i \leq n$ with weight χ_i

Thus, our resulting root system becomes $\{\pm e_i, \pm e_i \pm e_j | i > j\} \subset \mathbb{R}^n$. $\mathbf{O}^+(V, q)$ is simply connected (exercise to listener), and it has a corresponding adjoint group $\mathbf{Spin}(V, q)$. Thus, we arrive at the following classification of B_n:

Theorem 2 (Classification of B_n)**.** *Every simple group of type B_n is isomorphic to $\mathbf{O}^+(V, q)$ (simply connected) or $\mathbf{Spin}(V, q)$ (adjoint) where (V, q) is a F-vector space of dimension $2n+1$ with a regular quadratic form q.*

2.2.3 Type C

For type C and D I will skip some details and just give a brief overview of how we apply this method to these cases. Step 1: let V be a F-vector space of dimension $2n$ and h a nondegenerate alternating form on V. This allows us to construct the symplectic group $\mathbf{Sp}(V, h)$ embedded into $\mathbf{GL}_{2n}(F)$. Our torus $T = \text{diag}(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1})$. Step 2: we have $\text{Lie}(G)$ as all $x \in \text{End}(V)$ so that $h(xv, u) + h(v, xu) = 0$ for any $u, v \in V$. Then our weight subspaces are

1. Diagonal matrices with trivial weight
2. $F \cdot (E_{i,j+n} + E_{j,i+n})$ for $1 \leq i < j \leq n$ with weight $\chi_i \cdot \chi_j$
3. $F \cdot (E_{i+n,j} + E_{j+n,i})$ for $1 \leq i < j \leq n$ with weight $\chi_i^{-1} \cdot \chi_j^{-1}$
4. $F \cdot (E_{j+n,i+n} - E_{i,j})$ for $1 \leq i \neq j \leq n$ with weight $\chi_i \cdot \chi_j^{-1}$
5. $F \cdot E_{i,n+i}$ for $1 \leq i \leq n$ with weight χ_i^2
6. $F \cdot E_{n+i,i}$ for $1 \leq i \leq n$ with weight χ_i^{-2}

You may notice some similarities here to the characterisation of B_n. The language of Langlands duality explains to us that groups of type B_n are Langlands dual to groups of type C_n, and vice versa. In fact, G is simply connected if and only if ${}^L G$ is adjoint (and vice versa), so we have dual classifications between our simply connected B_n groups and adjoint C_n groups, and vice versa. Our root system is $\{\pm 2e_i, \pm e_i \pm e_j | i > j\}$. We arrive at the following classification:

Theorem 3 (Classification of C_n)**.** *A split simple group of type C_n is isomorphic to either $\mathbf{Sp}(V, h)$ (simply connected) or $\mathbf{PGSp}(V, h)$ (adjoint), where V is an F-vector space of dimension $2n$ and h is a nondegenerate alternating form.*

2.2.4 Type D

Type D is the most complicated of the classical groups because there can be several nontrivial subgroups of Λ/Λ_r. We proceed as before by taking V a F-vector space of dimension $2n$ and q a hyperbolic form on q. We consider $G = \mathbf{O}^+(V, q)$. The root system becomes $\{\pm e_i \pm e_j | i > j\}$. $\mathbf{O}^+(V, q)$ is simple if $n \geq 3$ and $\Lambda_r \subsetneq T^* \subsetneq \Lambda$. The corresponding simply connected group is $\mathbf{Spin}(V, q)$ and the adjoint group is $\mathbf{PGO}^+(V, q)$. When n is odd these are all, as Λ/Λ_r is cyclic, but when n is even, $\Lambda/\Lambda_r \simeq (\mathbb{Z}/2\mathbb{Z})^2$ so we have three groups in between, which correspond to $\mathbf{Spin}^{\pm}(V, q)$. Overall:

Theorem 4 (Classification of D_n)**.** *A split simple group of type D_n when n is odd is isomorphic to either $\mathbf{Spin}(V, q)$ (simply connected), $\mathbf{O}^+(V, q)$, or $\mathbf{PGO}^+(V, q)$ (adjoint), where V is an F-vector space of dimension $2n$ and q is a hyperbolic form. When n is even, the same previous set are possible, but there are also $\mathbf{Spin}^{\pm}(V, q)$.*