Spin 1/2

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1 Introduction

1.1 Spin

Angular momentum is a very important concept in classical mechanics. It is a conserved quantity and serves as a natural quantity of measure for systems that rotate (either about their centre of mass or about an external centre of rotation). In particular, there are two types of angular momentum: spin and orbital. Spin angular momentum, as the name implies, is angular momentum for an object rotating about its centre of mass, and orbital angular momentum is angular momentum for an object rotating about an external centre of rotation.

In quantum mechanics, both of these values still exist, but spin needs to be defined differently. We like to think of particles as "points," and as such, cannot spin in the literal sense – they cannot rotate about their centre of mass. So an interesting question is, how can we quantise spin? On the other hand, orbital angular momentum exists in exactly the same way and can be defined equivalently. It is far less interesting, and is not an *intrinsic* property of a particle the same way spin is, as it relies on an external frame of reference. So let's discuss spin in quantum mechanics.

As quantum mechanics was being developed, a key question was what exactly the analogy of spin angular momentum was, and how exactly it worked. In the early 1920s, physicists finally reached an understanding of the structure of atoms, and determined that atoms had a nucleus, consisting of several smaller particles, and electrons, which orbited around the nucleus. Niels Bohr formulated this first model which explained the *line spectrum* of a hydrogen atom via *space quantisation*: electrons can only orbit in fixed *orbitals*, the set of which is discrete. This discreteness would become an incredibly important fact in the development of quantum mechanics. Since atoms are not points, it makes sense to ask about its spin angular momentum, and in particular, to ask if the direction of the spin angular momentum (which we henceforth will call simply *spin*) is quantised. This was the Bohr-Sommerfield hypothesis.

In 1921, Otto Stern developed [Ste21] an experiment to test the Bohr-Sommerfield hypothesis. By heating up silver atoms in an electric furnace to create a flat beam of silver atoms, he proposed that the beam be sent through an inhomogeneous magnetic field, causing the atoms to deflect according to their spin. Classically, it would be expected that the silver atoms would deflect in a full spectrum of ways, corresponding to spin being a real-valued variable. But in 1922 Walther Gerlach was able to test Stern's experiment, and the experiment now known as the Stern-Gerlach experiment showed something quite fascinating: the silver atoms deflected in precisely two ways, not a full spectrum!

We now know that this is because all atoms have spin $\frac{1}{2}$, and thus have $2s + 1 = 2$ possible spin states, which this experiment was able to discriminate. This is part of why spin $\frac{1}{2}$ is so important: it describes all atoms, and thus all matter. Spin $\frac{1}{2}$ is the smallest nontrivial case, and thus what happens in this case is extremely important to understanding spin in general.

1.2 Spinors

Spinors are the mathematical formulation of spin $\frac{1}{2}$ particles. Intuitively, having half-integer spin means that particles transform by -1 after a rotation of 2π. So we can think of spinors as being attached to "double rotations" – a rotation alongside a "parity" indicator. Since $\mathrm{SO}(n)$ is the group of rotations of the n-sphere (aka the set of unit vectors in $\mathbb{R}^n$), the "spin group" should be the double cover of $\mathrm{SO}(n)$, which we call $\mathrm{Spin}(n)$:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 0$$

is exact for all n. For $n = 3$, for example, we have that $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$. Then a *spinor* is a vector that transforms under $\mathrm{Spin}(n)$, or more generally $\mathrm{Spin}(p, q)$, where $\mathrm{Spin}(p, q)$ is the double cover of $\mathrm{SO}(p, q)$, the special orthogonal group with respect to the bilinear form of signature (p, q). If V is a vector space with a bilinear form of signature (p, q) and $\rho : \mathrm{Spin}(p, q) \to \mathrm{GL}(V)$ is a spin representation on V, then an element of V is called a (p, q)-*spinor*. Spinors are extremely important because they allow us to define *spinor fields*, which are fields that take values in a spinor space (vector spaces with an action of $\mathrm{Spin}(p, q)$). Spinor fields are the canonical way to quantise particles with spin $\frac{1}{2}$, and serve as a very important starting point in understanding both the quantisation of matter and of particles with spin.

2 Lorentz Representations

In this paper we will take the $(+,-,-,-)$ convention, so the Minkowski form on $\mathbb{R}^4$ is $t^2 - x^2 - y^2 - z^2$ and the Lorentz group is $O(1,3)$.

2.1 Lorentz Transformations

When we work in $\mathbb{R}^{3+1}$, we know that our metric is not the usual Euclidean metric: instead, we have the *Minkowski metric*, given by $d(v,w) = c^2(v_t - w_t)^2 - (v_x - w_x)^2 - (v_y - w_y)^2 - (v_z - w_z)^2$. In natural units we can exclude the c^2 and simply write the bilinear form $g = \mathrm{diag}(1,-1,-1,-1)$, so that $g(v,w) = v^t \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} w$, and $d(v,w) = ||v-w||_g = g(v-w, v-w)^{1/2}$.

Einstein's *second postulate of relativity* tells us that in any local coordinate system, c should be the same. Explicitly, c is invariant of a choice of inertial coordinates. This means that any transformation of coordinates Λ must preserve the *spacetime interval* of a vector x, which is given by $||x||_g$. Thus for such a Λ, we have

$$\langle \Lambda x, \Lambda x \rangle_g = \langle x, x \rangle_g$$

So in particular, Λ preserves the bilinear form g. Since g is of signature $(1,3)$ (it has 1 positive and 3 negative eigenvalues), we write $O(1,3)$ (the *orthogonal group* of signature $(1,3)$) for the group of all such Λ. This is called the *Lorentz group*, and an element of it is called a *Lorentz transformation*. We will study how *Lorentz symmetry* underpins the study of particles with spin $\frac{1}{2}$.

Given a Lorentz transformation Λ, we have that $\det \Lambda = \pm 1$, and we call Λ *proper* if $\det \Lambda = 1$. The subgroup of proper Lorentz transformations is denoted $SO(1,3) \subset O(1,3)$. We can also define the subgroup $O(1,3)^c \subset O(1,3)$ of *orthochronous* (time-preserving) transformations: those with $\Lambda^0_0 \geq 1$. Then $O^0(1,3) = SO(1,3) \cap O(1,3)^c$ (where for a Lie group G we write G^0 for the connected component of the identity). When we discuss Lorentz invariance, we mean invariance under $O^0(1,3)$, the *restricted Lorentz group*, not the full Lorentz group. Parity ($\det \Lambda = -1$) and time reversal ($\Lambda^0_0 \leq -1$) are more complicated and need to be treated separately. For the remainder of this paper, we will write $\mathcal{L}$ for the restricted Lorentz group $O^0(1,3)$, and simply call it the Lorentz group.

Now, let's move to the setting of quantum field theory (following [Sre06] 2, 33). Let $\phi(x)$ be a field and Λ a Lorentz transformation. We can associate a unitary operator $U(\Lambda)$ to Λ that encapsulates Λ's role as a change of coordinates: $U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$. Then we have that

$$U(\Lambda)^{-1}\partial^\mu \phi(x) U(\Lambda) = \Lambda^\mu_\rho \overline{\partial}^\rho \phi(\Lambda^{-1}x)$$

where $\overline{x} := \Lambda^{-1}x$ is our new coordinates and $\overline{\partial} := \partial_{\overline{x}}$ is the derivative in this new frame. We can also define Lorentz transformations acting on *vector* and *tensor* fields:

$$U(\Lambda)^{-1}A^{\rho}(x)U(\Lambda) = \Lambda^{\mu}_{\rho}A^{\rho}(\Lambda^{-1}x) \qquad \text{(vector field)}$$
$$U(\Lambda)^{-1}B^{\mu\nu}(x)U(\Lambda) = \Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}B^{\rho\sigma}(\Lambda^{-1}x) \qquad \text{(tensor field)}$$

Note that if B is symmetric or antisymmetric, then that property is preserved by Lorentz transformations, and its trace $T(x) = g_{\mu\nu}B^{\mu\nu}(x)$ transforms like a scalar. So a natural question is, how can we split up B into components that have these preserved properties? Recall that any matrix can be uniquely written as a sum of a symmetric and antisymmetric component: $A_1 = \frac{A+A^t}{2}$ and $A_2 = \frac{A-A^t}{2}$ so that $A_1 + A_2 = A$ and $A_1^t = A_1, A_2^t = -A_2$. Further, since an antisymmetric matrix has zero diagonal, we can write D for the diagonal of A, so that $A_1' = A_1 - D$ has zero diagonal and $A = A_1' + D + A_2$, with A_1' symmetric and trace-zero, D diagonal, and A_2 antisymmetric. We can apply this to our B, getting

$$B^{\mu\nu}(x) = \underbrace{A^{\mu\nu}(x)}_{\text{antisymmetric}} + \underbrace{S^{\mu\nu}(x)}_{\text{symmetric, trace-zero}} + \underbrace{\frac{1}{4}g^{\mu\nu}T(x)}_{\text{diagonal}}$$

Importantly, since each property of being antisymmetric, symmetric, and diagonal is preserved under Lorentz transformations, this decomposition is also preserved by Lorentz transformations. And since the decomposition is unique, each component must be transformed to each component. So we say that these fields "don't mix" under Lorentz transformations.

But we don't know if these are the smallest possible fields that don't mix. We would like to understand the smallest possible fields that don't mix, which are exactly the *irreducible* representations of the Lorentz group. So let's investigate these.

2.2 Representations

Let $B^{\mu\nu}(x)$ be a tensor field and $\mathcal{H}$ the associated Hilbert space. Then by the previous part, we can write

$$U(\Lambda)^{-1}B^{\mu\nu}(x)U(\Lambda) = \Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}B^{\rho\sigma}(\Lambda^{-1}x)$$

and such a linear operator $U(\Lambda)$ exists for every $\Lambda \in \mathcal{L}$. In particular, we can require $U(\Lambda)$ to be unitary. Consider the map $U : \mathcal{L} \to \mathrm{GL}(\mathcal{H})$. This is a *representation* of the Lorentz group on $\mathcal{H}$: it is a linear homomorphism from $\mathcal{L}$ to $\mathrm{GL}(\mathcal{H})$. As $U(\Lambda)$ is unitary for all Λ, we have that U is actually a *unitary representation* of $\mathcal{L}$. So our previous problem can be described as classifying irreducible unitary representations of $\mathcal{L}$. It turns out that this problem cannot be solved in practice, as all nontrivial unitary representations of the Lorentz group are infinite-dimensional. So instead we will classify non-unitary representations and then see how we can get around the unitary condition (which will depend on the fact that our particles are fermionic).

First, we will pass to the lie algebra $\mathfrak{l} := \mathrm{Lie}(\mathcal{L}) = \mathfrak{so}(1,3)$. We only care about complex representations so we can freely pass to the complexification $\mathfrak{l}_{\mathbb{C}} = \mathfrak{so}(1,3)_{\mathbb{C}} = \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$. Clearly representations of $\mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$ can be indexed by pairs of representations of $\mathfrak{su}(2)_{\mathbb{C}}$, and these are well-known: they are in bijection with non-negative half-integers $\ell \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$ where each irreducible representation has dimension $2\ell+1$. So our irreducible Lorentz representations are of the form (ℓ, ℓ') with each ℓ, ℓ' a non-negative half-integer. Note that our numbering here agrees with *spin*, as spin n particles appear in the representation (ℓ, ℓ') with $\ell + \ell' = n$, while other sources (e.g. [Sre06] that we follow) number representations by their *dimensions*, i.e. $(2\ell+1, 2\ell'+1)$, while representation theorists number them by *highest weights*, i.e. $(2\ell, 2\ell')$. I normally use the highest-weight numbering but for the purpose of this paper we will use spin numbering.

However, there is a problem! The Lorentz group is not simply-connected, so not all of these representations are actually representations of the Lorentz group! In fact, if ℓ or ℓ' is half of an odd integer, then the representation (ℓ, ℓ') will not lift to the Lorentz group, but instead to its double cover, which is the $\mathrm{Spin}(1,3)$ group described in the introduction. We will allow these even though they aren't really representations of the Lorentz group. None of these representations are unitary, but we won't worry about this for now. When our dimension is 2, we call our spinors *Weyl spinors*. Specifically, if $(\ell, \ell') = (\frac{1}{2}, 0)$, we call the resulting representation *left-handed Weyl spinors* and $(0, \frac{1}{2})$ *right-handed Weyl spinors*.

We mentioned previously that $\mathfrak{spin}(1,3) \cong \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$. An interesting question is how the Hermitian conjugate acts on $\mathrm{Spin}(1,3) \cong \mathrm{SU}(2) \oplus \mathrm{SU}(2)$: we can write $\mathrm{Spin}(1,3) \cong \mathrm{SU}(2) \oplus i\,\mathrm{SU}(2)$ so that Hermitian conjugation permutes the factors $(a, b) \mapsto (b^\dagger, a^\dagger) = (b^{-1}, a^{-1})$ as a, b are unitary. Thus when we have a representation (ℓ, ℓ') of $\mathrm{Spin}(1,3)$, we have that $(\ell, \ell')^\dagger = (\ell', \ell)$, the dual representation to (ℓ', ℓ). Thus, the conjugate representation of (ℓ, ℓ') is (ℓ', ℓ). In particular, every right-handed spinor is precisely the conjugate of a left-handed spinor and vice-versa. So notationally, when ξ is a Weyl spinor, we will assume ξ is left-handed and $\xi^\dagger$ is right-handed. Every right-handed Weyl spinor is the conjugate of a left-handed Weyl spinor (i.e. its own conjugate, as $(\xi^\dagger)^\dagger = \xi$), so this is a valid choice to make.

2.3 Generators

When we explicitly define a representation, it is necessary to define a set of *generators* of our group. For the Lorentz group, we have three *rotations* and three *boosts*, for a total of 6 generators. We want to consider representations on the level of the Lie algebra, as before, where the Lorentz invariance instead becomes antisymmetry on the level of matrices. The space of antisymmetric 4-by-4 matrices has dimension 6, and we will label the basis matrices $M^{\sigma\rho}$ for $\sigma, \rho = 0, 1, 2, 3$, with the condition that $M^{\rho\sigma} = -M^{\sigma\rho}$, so there are only six relevant matrices. Then the relations on our Lie algebra will be

$$[M^{\mu\nu}, M^{\rho\sigma}] = g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} + g^{\mu\sigma}M^{\nu\rho} - g^{\nu\sigma}M^{\mu\rho} \tag{1}$$

For an arbitrary matrix $m \in \mathfrak{o}(1,3)$, we will write $m = \frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}$, so that $M = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$.

3 Dirac Equation

This approach on the Dirac equation follows [Ton06] Chapter 4.

3.1 Dirac Spinors

As we saw in the previous section, studying representations of the Lorentz group and spinors are very important in understanding how fields transform under Lorentz transformations. In order to understand spin $\frac{1}{2}$ fields, we will construct a specific representation of the Lorentz group that corresponds to all possible spin $\frac{1}{2}$ particles. Algebraically, we want the representation $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$, as all "simple" spin $\frac{1}{2}$ particles are either left-handed or right-handed Weyl spinors, so adding these together should produce all possible spin $\frac{1}{2}$ particles. We will call a *Dirac spinor* an element of the representation $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$, which is a four-component vector given by a pair of a left-handed and right-handed Weyl spinor. With this in mind, we can attempt to explicitly construct the representation of Dirac spinors.

To start constructing this representation, we will first define a representation of the *Clifford algebra*[1], which is an algebra satisfying $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\cdot I$. Explicitly, we need four matrices $\gamma^0,\gamma^1,\gamma^2,\gamma^3$ so that $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu\neq\nu$ and $(\gamma^0)^2 = I, (\gamma^i)^2 = -I$. Some basic linear algebra tells us that γ^μ must be of even size and of size greater than 2, so we can pick γ^μ to be 4-by-4 matrices. The classical choice is $\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$ and $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ (defined as 2-by-2 blocks, each of size 2-by-2 – this is because the Clifford algebra algebraically is of size 2^k when defined over a vector space of size k, which in this case is Weyl spinors), where σ^i are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now we can set $S^{\rho\sigma} := \frac{1}{4}[\gamma^\rho,\gamma^\sigma]$ (the genuine commutator, not an anticommutator as we wrote in the definition of the Clifford algebra). Then we claim that $[S^{\mu\nu},S^{\rho\sigma}] = g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} + g^{\mu\sigma}S^{\nu\rho} - g^{\nu\sigma}S^{\mu\rho}$, which means (via 1) that $S^{\mu\nu}$ form a representation of the Lorentz group. This is a four-dimensional representation of the Lorentz group, and we would hope that it is in fact $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$. We will soon see that this is in fact the case.

We want to define a field for $S^{\mu\nu}$ to act on. So let $\psi^\alpha(x)$ be a Dirac spinor field, which is to say it has four components, corresponding to $\alpha = 1,2,3,4$, each of which is complex. So explicitly $\psi^\alpha(x)$ takes values in $\mathbb{C}^4$, but we will understand $\mathbb{C}^4$ to actually be the Dirac representation of the Lorentz group, so we say $\psi^\alpha(x)$ takes values in Dirac spinors. Then under Lorentz transformations we have $\psi^\alpha(x) \mapsto S[\Lambda]^\alpha_\beta\psi^\beta(\Lambda^{-1}x)$, where

[1] Which is also the first step in defining a vertex algebra associated to a gauge theory.

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$$
$$S[\Lambda] := \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)$$

(that is, we define the representation explicitly by sending the generators $M^{\rho\sigma} \mapsto S^{\rho\sigma}$). Since we have explicit matrices for γ^μ, we can write explicit matrices for $S[\Lambda]$ for some choices of Λ. In particular, if Λ is a rotation,

$$S[\Lambda] = \begin{pmatrix} e^{i\phi\cdot\sigma/2} & 0 \\ 0 & e^{i\phi\cdot\sigma/2} \end{pmatrix}$$

and for a boost

$$S[\Lambda] = \begin{pmatrix} e^{\chi\cdot\sigma/2} & 0 \\ 0 & e^{-\chi\cdot\sigma/2} \end{pmatrix}$$

So the Dirac representation of the Lorentz group decomposes as a sum of two representations. We can explicitly check that these two representations are the two Weyl representations, so that $\phi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$, where u_+ is a left-handed Weyl spinor and u_- is a right-handed Weyl spinor. So our representation is exactly the one we want, and we can proceed.

Now, let's construct a Lorentz invariant equation of motion, stemming from a Lorentz invariant action. Let $\overline{\psi}(x) := \psi^\dagger(x)\gamma^0$ be the *Dirac adjoint* to $\psi(x)$. Then $\overline{\psi}\psi$ is a Lorentz scalar (i.e. transforms like a scalar, à la Section 2.1), $\overline{\psi}\gamma^\mu\psi$ is a Lorentz vector, and $\overline{\psi}\gamma^\mu\gamma^\nu\psi$ is a Lorentz tensor. Then we can define

$$\mathscr{L} = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \tag{2}$$
$$\begin{aligned} L &= \int \mathscr{L}\, d^4x \\ &= \int \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)\, d^4x \end{aligned} \tag{3}$$

Then our equation of motion (via Euler-Lagrange) becomes

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

This is the Dirac equation, an incredibly important and fascinating equation. For simplicity of notation, we will define the Feynman slash $\not{A} := A_\mu\gamma^\mu$, so that the Dirac equation becomes

$$(i\not{\partial} - m)\psi = 0$$

3.2 Plane Wave Solutions

Following [Ton06] 4.7.

We want to find solutions to the Dirac equation $(i\partial\!\!\!/ - m)\psi = 0$. Using the ansatz $\psi = u(p)e^{-ip\cdot x}$ (a usual ansatz for such equations), where u is a Dirac spinor depending only on momentum, we then have

$$(p\!\!\!/ - m)u(p) = 0$$

The solution, which we call the *positive-frequency solution*, is then $u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\overline{\sigma}}\xi \end{pmatrix}$, where ξ is a Weyl spinor such that $\xi^\dagger\xi = 1$.

Proof. We can rewrite the Dirac equation as

$$\begin{aligned}
(\gamma^\mu p_\mu - m)u(p) &= \begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\overline{\sigma}^\mu & -m \end{pmatrix} u(p) \\
&= \begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\overline{\sigma}^\mu & -m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
&= \begin{pmatrix} -mu_1 + p_\mu\sigma^\mu u_2 \\ p_\mu\overline{\sigma}^\mu u_1 - mu_2 \end{pmatrix} \\
&= \begin{pmatrix} (p\cdot\sigma)u_2 - mu_1 \\ (p\cdot\overline{\sigma})u_1 - mu_2 \end{pmatrix} \\
&= 0
\end{aligned}$$

for u_1, u_2 two-component vectors. So $(p\cdot\sigma)u_2 = mu_1, (p\cdot\overline{\sigma})u_1 = mu_2$. Since $(p\cdot\sigma)(p\cdot\overline{\sigma}) = m^2$ these equations are actually identical. As an ansatz, let $u_1 = (p\cdot\sigma)\xi'$ for ξ' a Weyl spinor. Then $u_2 = m\xi'$, so

$$u(p) = \alpha\begin{pmatrix} (p\cdot\sigma)\xi' \\ m\xi' \end{pmatrix}$$

for a scalar α. Since $(p\cdot\sigma)(p\cdot\overline{\sigma}) = m^2$, $\sqrt{p\cdot\sigma}\sqrt{p\cdot\overline{\sigma}} = m$, so letting $\alpha = (\sqrt{p\cdot\sigma})^{-1}$ gives us

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\overline{\sigma}}\xi \end{pmatrix}$$

for $\xi = \sqrt{p\cdot\sigma}\xi'$, as we wanted. $\square$

We can also take the opposite ansatz $\psi = v(p)e^{ip\cdot x}$ and arrive at a very similar solution $v(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta \\ -\sqrt{p\cdot\overline{\sigma}}\eta \end{pmatrix}$, which we call the *negative-frequency solution*, where η is a Weyl spinor such that $\eta^\dagger\eta = 1$. These two solutions $u(p)$ and $v(p)$ will be incredibly important in our quantisation in the next section. These $u(p)$ and $v(p)$ are in fact *Dirac spinors* (which is clear by their definition), and are the prototypical examples of Dirac spinors that we will use to quantise Dirac fields.

Inner Products

In order to make computations easier, we will introduce a basis $\xi^s, s = 1, 2$ so that $\xi^{r\dagger}\xi^s = \delta^{rs}$, and similar for η^s. Then $u^s(p) := \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\overline{\sigma}}\xi^s \end{pmatrix}$ so that a general positive-frequency solution is given by $a_s u^s$ for a_s scalars. We define $v^s(p)$ similarly. Then we would like to compute various inner products, $u^\dagger\cdot u, \overline{u}\cdot u$, etc.

$$
\begin{aligned}
u^{r\dagger}(p)\cdot u^s(p) &= \begin{pmatrix} \xi^{r\dagger}\sqrt{p\cdot\sigma} & \xi^{r\dagger}\sqrt{p\cdot\overline{\sigma}} \end{pmatrix} \begin{pmatrix} \xi^s\sqrt{p\cdot\sigma} \\ \xi^s\sqrt{p\cdot\overline{\sigma}} \end{pmatrix} \\
&= \xi^{r\dagger}(p\cdot\sigma)\xi^s + \xi^{r\dagger}(p\cdot\overline{\sigma})\xi^s \\
&= 2\xi^{r\dagger}p_0\xi^s \\
&= 2p_0\delta^{rs} \\
\overline{u}^r(p)\cdot u^s(p) &= \begin{pmatrix} \xi^{r\dagger}\sqrt{p\cdot\sigma} & \xi^{r\dagger}\sqrt{p\cdot\overline{\sigma}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi^s\sqrt{p\cdot\sigma} \\ \xi^s\sqrt{p\cdot\overline{\sigma}} \end{pmatrix} \\
&= 2m\delta^{rs}
\end{aligned}
$$

and similar for v. For mixed inner products, we compute

$$
\begin{aligned}
\overline{u}^r(p)\cdot v^s(p) &= \begin{pmatrix} \xi^{r\dagger}\sqrt{p\cdot\sigma} & \xi^{r\dagger}\sqrt{p\cdot\overline{\sigma}} \end{pmatrix} \gamma^0 \begin{pmatrix} \eta^s\sqrt{p\cdot\sigma} \\ -\eta^s\sqrt{p\cdot\overline{\sigma}} \end{pmatrix} \\
&= \xi^{r\dagger}\sqrt{(p\cdot\sigma)(p\cdot\overline{\sigma})}\eta^s - \xi^{r\dagger}\sqrt{(p\cdot\sigma)(p\cdot\overline{\sigma})}\eta^s \\
&= 0
\end{aligned}
$$

and we similarly compute $u^{r\dagger}(p)\cdot v^s(-p) = 0$. So in conclusion we have

$$
\begin{aligned}
u^{r\dagger}(p)\cdot u^s(p) &= 2p_0\delta^{rs} && (4) \\
\overline{u}^r(p)\cdot u^s(p) &= 2m\delta^{rs} && (5) \\
v^{r\dagger}(p)\cdot v^s(p) &= 2p_0\delta^{rs} && (6) \\
\overline{v}^r(p)\cdot v^s(p) &= -2m\delta^{rs} && (7) \\
\overline{u}^r(p)\cdot v^s(p) &= \overline{v}^r(p)\cdot u^s(p) = 0 && (8) \\
u^{r\dagger}(p)\cdot v^s(-p) &= v^{r\dagger}(p)\cdot u^s(-p) = 0 && (9)
\end{aligned}
$$

Outer Products

We claim that

$$
\begin{aligned}
u^s(p)\overline{u}^s(p) &= \not{p} + m && (10) \\
v^s(p)\overline{v}^s(p) &= \not{p} - m && (11)
\end{aligned}
$$

Proof. We will prove the result for u; the result for v is proven similarly.

$$\begin{aligned}
u^s(p)\overline{u}^s(p) &= \begin{pmatrix} \xi^s\sqrt{p\cdot\sigma} \\ \xi^s\sqrt{p\cdot\overline{\sigma}} \end{pmatrix} \begin{pmatrix} \xi^{s\dagger}\sqrt{p\cdot\overline{\sigma}} & \xi^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix} \\
&= \begin{pmatrix} \xi^s\xi^{s\dagger}m & \xi^s\xi^{s\dagger}p\cdot\sigma \\ \xi^s\xi^{s\dagger}p\cdot\overline{\sigma} & \xi^s\xi^{s\dagger}m \end{pmatrix} \\
&= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\overline{\sigma} & m \end{pmatrix} && (\xi^s\xi^{s\dagger} = I_2) \\
&= \not{p} + mI_4 = \not{p} + m
\end{aligned}$$

$\square$

With these formuale in hand, we are now ready to start quantising our Dirac field!

4 Quantisation of Dirac Fields

This approach to quantising the Dirac field follows [Ton06] Chapter 5. As a style note, we use math script ($\mathscr{L}, \mathscr{H}$) rather than math calligraphy ($\mathcal{L}, \mathcal{H}$) for the Lagrangian and Hamiltonian as $\mathcal{L}$ is already used for the Lorentz group and $\mathcal{H}$ for the Hilbert space.

4.1 First Steps

Now that we have our Lagrangian $\mathscr{L} = \overline{\psi}(x)(i\partial\!\!\!/ - m)\psi(x)$ (2), we would like to quantise it. If we attempt to quantise $\mathscr{L}$ in the usual way (i.e. for a bosonic field), we will quickly run into issues! Let's try it and see what goes wrong.

We start as usual by defining the momentum $\pi = \frac{\partial \mathscr{L}}{\partial \dot{\psi}} = i\overline{\psi}\gamma^0 = i\psi^\dagger$. It seems strange that momentum isn't a time derivative, but this is in order for the equations of motion to be first order in time ([Ton06] 5.2). Then, again as usual, we write the canonical commutation relations:

$$
\begin{aligned}
[\psi_a(x), \psi_b(x)] &= 0 \\
[\psi_a^\dagger(x), \psi_b^\dagger(x)] &= 0 \\
[\psi_a(x), \psi_b^\dagger(x)] &= \delta_{ab}\delta(x-y)
\end{aligned}
$$

But as we will quickly see, the third equation cannot be consistent with spin $\frac{1}{2}$! Explicitly, we have erroneously assumed that our elements are *even*, i.e. bosonic, when they are in fact *odd*, i.e. fermionic, à la the Spin-Statistics theorem. The terminology "even" and "odd" refers to the grading of a super-vertex-algebra structure, in which the commutator of *even* elements is the usual bracket $[A, B] = AB - BA$ and the commutator of *odd* elements is the anticommutator bracket $[A, B]_+ = AB + BA$. So let's see exactly why spin $\frac{1}{2}$ cannot be bosonic (and in a sense, prove a very small part of the Spin-Statistics theorem).

As we have just defined a free bosonic theory, we can write the solution in plane waves as we already know how to do:

$$
\begin{aligned}
\psi(x) &= \sum_{s=1,2} \int \left[b_p^s u^s(p) e^{ip\cdot x} + c_p^{s\dagger} v^s(p) e^{-ip\cdot x} \right] \widetilde{dp} \\
\psi^\dagger(x) &= \sum_{s=1,2} \int \left[b_p^{s\dagger} u^s(p) e^{ip\cdot x} + c_p^{s} v^s(p) e^{-ip\cdot x} \right] \widetilde{dp}
\end{aligned}
$$

where $\widetilde{dp} := \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}}$ for ω_p a frequency we will not write out, and u^s, v^s are the spinors we previously described.

Lemma 1. *We must have*

$$
\begin{aligned}
[b_p^r, b_q^{s\dagger}] &= (2\pi)^3 \delta^{rs} \delta(p-q) \\
[c_p^r, c_q^{s\dagger}] &= -(2\pi)^3 \delta^{rs} \delta(p-q)
\end{aligned} \tag{12}
$$

And all other brackets vanishing.

See [Ton06] for the proof of this.

The minus sign appearing in the second equation is going to be a problem! We can introduce the Hamiltonian $\mathscr{H} = \pi\dot{\psi} - \mathscr{L} = \overline{\psi}(-i\gamma^i \partial_i + m)\psi$. To quantise $H = \int \mathscr{H}\, d^3x$, we start by considering

$$
(-i\gamma^i \partial_i + m)\psi = \sum_{s=1,2} \int \left[b_p^s(-\gamma^i p_i + m) u^s(p) e^{ip\cdot x} + c_p^{s\dagger}(\gamma^i p_i + m) v^s(p) e^{-ip\cdot x} \right] \widetilde{dp}.
$$

But $(-\gamma^i p_i + m)u^s(p) = \gamma^0 p_0 u^s(p)$ and $(\gamma^i p_i + m)v^s(p) = -\gamma^0 p_0 v^s(p)$, so

$$
(-i\gamma^i \partial_i + m)\psi = \sum_{s=1,2} \int \omega_p \gamma^0 \left[b_p^s u^s(p) e^{ip\cdot x} - c_p^{s\dagger} v^s(p) e^{-ip\cdot x} \right] \widetilde{dp}.
$$

Then

$$
\begin{aligned}
H &= \int \psi^\dagger (-i\gamma^i \partial_i + m)\psi \, d^3x \\
&= \int \omega_p \left[b_q^{r\dagger} u^r(q)^\dagger e^{-iq\cdot x} + c_q^r v^{r\dagger}(q) e^{iq\cdot x} \right] \cdot \left[b_p^s u^s(p) e^{ip\cdot x} - c_p^{s\dagger} v^s(p) e^{-ip\cdot x} \right] d^3x \, \widetilde{dp}\, \widetilde{dq} \\
&= \frac{1}{2(2\pi)^3} \int \Big(b_p^r b_p^s \left[u^r(p)^\dagger \cdot u^s(p) \right] - c_p^r c_p^{s\dagger} \left[v^r(p)^\dagger \cdot v^s(p) \right] \\
&\quad - b_p^{r\dagger} c_{-p}^{s\dagger} \left[u^r(p)^\dagger \cdot v^s(-p) \right] + c_p^r b_{-p}^s \left[v^r(p)^\dagger \cdot u^s(-p) \right] \Big) \, d^3p
\end{aligned}
$$

Then by 4, 6, 8, 9 we have

$$
\begin{aligned}
H &= \int b_p^{s\dagger} b_p^s - c_p^s c_p^{s\dagger} \, \widetilde{dp} \\
&= \int b_p^{s\dagger} b_p^s - c_p^{s\dagger} c_p^s + (2\pi)^3 \delta(0) \, \widetilde{dp} && \text{(Using 12)}
\end{aligned}
$$

The $\delta(0)$ can be ignored using normal ordering, but the minus sign in front of $c_p^{s\dagger} c_p^s$ is concerning. If we take $c^\dagger$ to be our creation operators, then $[H, c_p^{s\dagger}] = \omega_p c_p^{s\dagger}$ which have negative norm from 12, so this doesn't work. So instead we must have c be the creation operators. But $[H, c_s^p] = -\omega_p c_p^s$, which means that we can produce infinitely many c particles, as each costs negative energy! So no matter which way we assign things, a minus sign appears which ruins things. How can we fix this?

4.2 Fermionic (Odd) Elements

As we knew *a posteriori*, by the Spin-Statistics theorem, we can't assign a bosonic quantisation to a particle with spin $\frac{1}{2}$. Instead, we need to assign *fermionic*, i.e. Fermi-Dirac, statistics. Explicitly, we need to replace 12 with *anticommutators*[2]:

$$\begin{aligned}
\{\psi_a(x), \psi_b(x)\} &= 0 \\
\{\psi_a^\dagger(x), \psi_b^\dagger(x)\} &= 0 \\
\{\psi_a(x), \psi_b^\dagger(x)\} &= \delta_{ab}\delta(x-y) \\
\{b_p^r, b_q^{s\dagger}\} &= (2\pi)^3\delta^{rs}\delta(p-q) \\
\{c_p^r, c_q^{s\dagger}\} &= (2\pi)^3\delta^{rs}\delta(p-q)
\end{aligned} \tag{13}$$

where $\{A, B\}$ is the *anticommutator* $\{A, B\} = AB+BA$. Then our Hamiltonian computation goes identically except

$$\begin{aligned}
H &= \int b_p^{s\dagger} b_p^s - c_p^s c_p^{s\dagger}\, \widetilde{dp} \\
&= \int b_p^{s\dagger} b_p^s + c_p^{s\dagger} c_p^s - (2\pi)^3\delta(0)\, \widetilde{dp} && \text{(Using 13)}
\end{aligned}$$

Which resolves our problems. We also can compute[3]

$$\begin{aligned}
[H, b_p^r] &= -\omega_p b_p^r \\
[H, b_p^{r\dagger}] &= \omega_p b_p^{r\dagger} \\
[H, c_p^r] &= -\omega_p c_p^r \\
[H, c_p^{r\dagger}] &= \omega_p c_p^{r\dagger}
\end{aligned}$$

So everything will work as expected and no infinite stream of negative-energy particles will appear. We can define $|0\rangle$ in the usual way so that $b_p^s|0\rangle = c_p^s|0\rangle = 0$, and then we can construct the Fock space[4]. We have our one-particle states $|p, r\rangle := b_p^{r\dagger}|0\rangle$ and two-particle states $|p, r; p', r'\rangle := b_p^{r\dagger} b_{p'}^{r'\dagger}|0\rangle$

[2]An equivalent way of stating this solution on the level of vertex algebras is that our vertex algebra is actually a *superalgebra* and fermions/fermionic creation and annihilation operators lie in the odd component. This is the first step in stating the boson-fermion correspondence.

[3]This is the genuine *commutator* as H is even and the pairing of an even and odd element is a commutator. We also produce an odd element as we would expect pairing an even and odd element.

[4]In the vertex-algebra sense, given a vertex algebra V and chosen subalgebra $V_{\geq 0}$, the Fock representation is $\mathrm{Ind}_{V_{\geq 0}}^{V}\mathbb{C}$, the induced representation of the trivial representation from $V_{\geq 0}$ to V. In our case, V is our vertex algebra generated by all our creation and annihilation operators and $V_{\geq 0}$ is the subalgebra generated by only annihilation operators. This is a standard construction in the theory of vertex algebras.

that satisfy $|p,r;p',r'\rangle = -|p;,r';p,r\rangle$ as $\{b_p^{r\dagger}, b_{p'}^{r'\dagger}\} = 0$. Thus Fermi-Dirac statistics are satisfied, and we arrive at the Pauli exclusion principle as an immediate corollary: $|p,r;p,r\rangle = -|p,r;p,r\rangle = 0$[5]. We can also compute $J|0,r\rangle$ (the angular momentum operator) indeed acts by angular momentum $\frac{1}{2}$.

4.3 Propagators

Now, let us switch to the Heisenberg model. Specifically, we want to define a spinor-valued field $\psi(x,t)$ so that

$$\frac{\partial\psi}{\partial t} = i[H,\psi]$$

We can explicitly solve this via plane-wave expansion like usual:

$$\psi(x) = \int b_p^s u^s(p)e^{-ip\cdot x} + c_p^{s\dagger} v^s(p)e^{ip\cdot x}\,\widetilde{dp}$$
$$\psi^\dagger(x) = \int b_p^{s\dagger} u^s(p)^\dagger e^{-ip\cdot x} + c_p^{s} v^s(p)^\dagger e^{ip\cdot x}\,\widetilde{dp}$$

Then we define $iS_{\alpha\beta}(x-y) := \{\psi_\alpha(x), \overline{\psi}_\beta(x)\}$, the *fermionic propagator*. We will just write $iS(x-y)$ and omit the indices.

$$\begin{aligned} iS(x-y) &= \int \left[\{b_p^s, b_q^{r\dagger}\}u^s(p)\overline{u}^r(q)e^{-i(p\cdot x - q\cdot y)} + \{c_p^{s\dagger}, c_q^r\}v^s(p)\overline{v}^r(q)e^{i(p\cdot x-q\cdot y)}\right]\widetilde{dp}\,\widetilde{dq} \\ &= \int \frac{1}{\sqrt{2\omega_p}}\left[u^s(p)\overline{u}^s(p)e^{-ip\cdot(x-y)} + v^s(p)\overline{v}^s(p)e^{ip\cdot(x-y)}\right]\widetilde{dp} \\ &= \int \frac{1}{\sqrt{2\omega_p}}(\not{p}+m)e^{-ip\cdot(x-y)} + (\not{p}-m)e^{ip\cdot(x-y)}\,\widetilde{dp} \quad \text{(Using 10, 11)} \end{aligned}$$

So $iS(x-y) = (i\not{\partial}_x + m)(D(x-y) - D(y-x))$, where

$$D(x-y) = \int \frac{1}{\sqrt{2\omega_p}}e^{-ip\cdot(x-y)}\,\widetilde{dp}$$

is this usual propagator of a real scalar field ([Ton06] 5.28).

Finally, we can describe the Feynman propagator and Green's function. First, we calculate the vacuum expectation values

$$\langle 0|\,\psi_\alpha(x)\overline{\psi}_\beta(y)\,|0\rangle = \int \frac{1}{\sqrt{2\omega_p}}(\not{p}+m)_{\alpha\beta}e^{-ip\cdot(x-y)}$$
$$\langle 0|\,\overline{\psi}_\beta(x)\psi_\alpha(y)\,|0\rangle = \int \frac{1}{\sqrt{2\omega_p}}(\not{p}-m)_{\alpha\beta}e^{ip\cdot(x-y)}$$

[5] This is again immediately understood by considering that fermions are odd, so they anticommute.

Then $S_F(x-y) := \langle 0| T\psi(x)\overline{\psi}(y)|0\rangle$, and we can write it as

$$S_F(x-y) = i\int e^{-ip\cdot(x-y)}\frac{\gamma\cdot p+m}{p^2-m^2+i\epsilon}\frac{d^4p}{(2\pi)^4}$$

which indeed satisfies $(i\not{\partial}_x - m)S_F(x-y) = i\delta(x-y)$. To find all such Green's functions, we proceed identically to the bosonic case (where we already know how to do this), except commutation in both normal-ordered and time-ordered products becomes anticommutation. With this simple modification, Wick's theorem proceeds as normal. And thus, we have fully quantised our spin $\frac{1}{2}$ field.

References

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