McKay Correspondence II

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Reflection Functors and Coxeter Elements

Briefly, let us recall the tools of reflection functors and Coxeter functors. Let $\vec{Q}$ be a quiver with vertices I, and denote s_r for the simple reflection corresponding to r, for r a simple real root. We have a *Coxeter element* $C = \prod_R s_r$ associated to a choice of ordering of R, our set of simple real roots. We have a *reflection functor* $\Phi_r^{\pm} : \mathrm{Rep}(\vec{Q}) \to \mathrm{Rep}(s_r^{\pm}\vec{Q})$, and define the *Coxeter functor* as $\mathbf{C}^{\pm} = \prod_R \Phi_r^{\pm}$.

Preprojective and Preinjective Representations

Let V be an indecomposable representation of $\vec{Q}$. We say V is

1. *preprojective* if $(\mathbf{C}^+)^n V = 0, n \gg 0$
2. *preinjective* if $(\mathbf{C}^-)^n V = 0, n \gg 0$
3. *regular* if $(\mathbf{C}^+)^n V \neq 0, n > 0$

(Note that $\mathbf{C}^\pm V$ is indecomposable when V is, and $\mathbf{C}^\pm V = 0$ iff V is projective, respectively injective.)

Translation Quiver and Slices

Recall that for any quiver Q, we have a quiver $\mathbb{Z}Q$ called the *translation quiver*, defined by $\mathbb{Z}Q := \{(i,n)|p(i)+n \equiv 0 \mod 2\} \subset Q \times \mathbb{Z}$. We will temporarily ignore what $p(i)$ is defined as – when we use our translation quivers, this will be clear.

We say $T \subset \mathbb{Z}Q$ is a *slice* if $\forall i \in I$, $\exists! q = (i, h_i) \in T$ and when i, j are connected by an edge in Q, $h_i = h_j \pm 1$. We previously detailed that given a slice T, we obtain an orientation of Q, denoted by $\vec{Q}_T$, where $e : i \to j$ if $h_i = h_j + 1$ and $e : j \to i$ if $h_i + 1 = h_j$.

Setup

Let $G \leq \mathrm{SU}(2)$ be a finite subgroup and Q be the corresponding Euclidean graph we introduced last week. For simplicity assume $-I \in G$, so that $\overline{G} := \pi(G)$ gives us $G = \pi^{-1}(\overline{G})$.
Pick some $X \in \mathrm{Rep}(G)$. On X, $-I$ acts as either I or $-I$: if $-I$ acts as I, set $p(X) := 0$, and if $-I$ acts as $-I$, set $p(X) = 1$.
This introduces a $\mathbb{Z}/2\mathbb{Z}$ grading on $\mathrm{Rep}(G)$: let $\mathrm{Rep}_0(G) := \{X \in \mathrm{Rep}(G) | p(X) = 0\}$ and similar for $\mathrm{Rep}_1(G)$.
Then $\mathrm{Rep}(G) = \mathrm{Rep}_0(G) \oplus \mathrm{Rep}_1(G)$. We call $p(X)$ the *parity* of X.

Equivariant Sheaves

Let X be a scheme. A *G-equivariant coherent sheaf* on X is the data of a coherent sheaf $\mathcal{F}$ on X together with an isomorphism $\phi : \sigma^*\mathcal{F} \to p_2^*\mathcal{F}$, where σ is the action map $G \times X \to X$ and p_2 is the projection $G \times X \to X$ that satisfies the *cocycle condition*:

$$p_{23}^*\phi \circ (1_G \times \sigma)^*\phi = (m \times 1_X)^*\phi$$

where m represents multiplication $G \times G \to G$ and the $p_{(-)}$ are the projections in $G \times G \times X$.

Equivariant Sheaves II

Let $\mathrm{Coh}_G(\mathbb{P}^1)$ be the category of G-equivariant coherent sheaves on $\mathbb{P}^1$. Similarly, $\mathrm{Coh}_{\overline{G}}(\mathbb{P}^1)$ is the category of $\overline{G}$-equivariant coherent sheaves on $\mathbb{P}^1$. We can describe $\overline{G}$-equivariant sheaves in terms of G-equivariant sheaves:

$$\mathrm{Coh}_{\overline{G}}(\mathbb{P}^1) = \{\mathcal{F} \in \mathrm{Coh}_G(\mathbb{P}^1) | (-I)^*|_{\mathcal{F}} = \mathrm{id}\}$$

Note that for $\mathcal{F} \in \mathrm{Coh}_G(\mathbb{P}^1)$, $H^0(\mathbb{P}^1, \mathcal{F}) = \Gamma(\mathcal{F})$ has the natural structure of a G-representation.

Example

Example

$\mathcal{O}(n)$ has a natural structure of a G-equivariant sheaf for any $n \in \mathbb{N}$. Since $(-I)^*|_{\mathcal{O}(n)} = (-1)^n$, we see $\mathcal{O}(n)$ is $\overline{G}$-equivariant iff n is even.
When X is a representation of G, $X(n) := X \otimes \mathcal{O}(n)$ is a G-equivariant locally free sheaf, and $\overline{G}$-equivariant when $p(X) + n$ is even.

Some Homological Facts

Theorem ([Kir16] 8.20)

Let $\mathcal{C} = \mathrm{Coh}_G\left(\mathbb{P}^1\right)$. Then:

1. *$\mathcal{C}$ is hereditary: for any $\mathcal{F}, \mathcal{G} \in \mathcal{C}$, we have $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ for $i > 1$.*
2. *Serre duality: if $\mathcal{F}, \mathcal{G}$ are locally free, then we have an isomorphism*

$$\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{F}, \mathcal{G}(-2)) = \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{F}(2), \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, \mathcal{F})^*.$$

Some Homological Facts II

3. For any locally free sheaf $\mathcal{F} \in \mathcal{C}$, we have a short exact sequence

$$0 \to \mathcal{F} \to \rho \otimes \mathcal{F}(1) \to \Lambda^2 \rho \otimes \mathcal{F}(2) \simeq \mathcal{F}(2) \to 0,$$

   where $\rho = \Gamma(\mathcal{O}(1)) \simeq \mathbb{C}^2$ is the standard two-dimensional representation of G. (Note that $\rho \simeq \rho^*$.)

4. Every G-equivariant coherent sheaf admits a resolution which consists of locally free G-equivariant sheaves. Every locally free G equivariant sheaf is a direct sum of sheaves of the form $X \otimes \mathcal{O}(n)$.

Categories of Representations

Let us assign to every vertex $q = (i, n)$ of $\mathbb{Z}Q$ a locally free $\bar{G}$-equivariant sheaf on $\mathbb{P}^1$ by

$$X_q = \rho_i \otimes \mathcal{O}(n), \quad q = (i, n), i \in I, n \in \mathbb{Z},$$

where ρ_i is the irreducible representation of G corresponding to $i \in I$. Note that then

$$X_{\tau q} = X_{(i,n-2)} = X_q(-2)$$

Since the edges of Q correspond to morphisms $\rho_i \to \rho_j \otimes \rho$, we get, for every edge $h : i \to j$ in Q, a morphism

$$x_h : X_{(i,n)} = \rho_i \otimes \mathcal{O}(n) \to \rho_j \otimes \rho \otimes \mathcal{O}(n) \to \rho_j \otimes \mathcal{O}(n+1) = X_{(j,n+1)},$$

where the morphism $\rho \otimes \mathcal{O}(n) \to \mathcal{O}(n+1)$ is constructed using the isomorphism $\rho \simeq \Gamma(\mathcal{O}(1))$.

Thus, the collection of sheaves $X_q, q \in \mathbb{Z}Q$, forms a representation of $\mathbb{Z}Q$ (in $\mathcal{C}$).

Slices

Lemma ([Kir16] 8.21)

Let $T \subset \mathbb{Z}Q$ be a slice. Denote $\mathcal{C} = \mathrm{Coh}_{\bar{G}}\left(\mathbb{P}^1\right)$ and let $D^b_{\bar{G}}\left(\mathbb{P}^1\right)$ be the corresponding derived category: $D^b_{\bar{G}}\left(\mathbb{P}^1\right) = D^b(\mathcal{C})$.

1. *Sheaves $X_q, q \in T$, generate $D^b_{\bar{G}}\left(\mathbb{P}^1\right)$ as a triangulated category: the smallest triangulated subcategory in $D^b_{\bar{G}}\left(\mathbb{P}^1\right)$ containing all X_q is $D^b_{\bar{G}}\left(\mathbb{P}^1\right)$.*
2. *If $q \in T, p \prec T$, then $\mathrm{Hom}_{\mathcal{C}}\left(X_q, X_p\right) = 0$. Similarly, if $p \succcurlyeq T$, then $\mathrm{Ext}^1_{\mathcal{C}}\left(X_q, X_p\right) = 0$*
3. *If $p, q \in T$, then*

$$\mathrm{Hom}_{\mathcal{C}}\left(X_q, X_p\right) = \langle \textit{paths in } T \textit{ from } q \textit{ to } p \rangle,$$
$$\mathrm{Ext}^1_{\mathcal{C}}\left(X_q, X_p\right) = 0.$$

Slice Functors

Definition (slice functor, [Kir16] 8.22)

Let $T = \{(i, h_i)\} \subset \mathbb{Z}Q$ be a slice. We define the functor

$$\Psi_T : \mathrm{Coh}_{\bar{G}}\left(\mathbb{P}^1\right) \to \mathrm{Rep}\left(\vec{Q}_T\right)$$

by

$$\Psi_T(\mathcal{F}) = \bigoplus_{i \in I} \mathrm{Hom}_{\mathcal{C}}\left(X_{(i,h_i)}, \mathcal{F}\right)$$

Slice Functors II

and the maps corresponding to edges of $\vec{Q}_T$ are given by

$$\mathrm{Hom}_{\mathcal{C}}\left(X_{(i,h_i)}, \mathcal{F}\right) \to \mathrm{Hom}_{\mathcal{C}}\left(X_{(j,h_j)}, \mathcal{F}\right), \quad h_i = h_j + 1,$$
$$f \mapsto f \circ x_{\tilde{e}}$$

where e is an edge between i and j in Q (and thus an edge $i \to j$ in $\vec{Q}_T$), $\tilde{e} : (j, h_j) \to (i, h_j + 1)$ is the corresponding edge in $\mathbb{Z}Q$.

Example

Example ([Kir16] 8.23)

Let $\mathcal{F} = X_p, p = (i, h_i) \in T$. Then it follows from the previous lemma that

$$\Psi_T(X_p) = P(i)$$

is the standard projective representation of $\vec{Q}_T$ we have discussed.

Important Theorem

Theorem ([Kir16] 8.24)

Let $G \subset \mathrm{SU}(2)$ be a finite subgroup containing $-I$. Let Q be the corresponding Euclidean graph, and let $T \subset \mathbb{Z}Q$ be a slice.

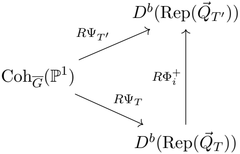
1. *The functor $\Psi_T : \mathrm{Coh}_{\bar{G}}\left(\mathbb{P}^1\right) \to \mathrm{Rep}\left(\vec{Q}_T\right)$ is left exact.*
2. *The derived functor*

$$R\Psi_T : D^b_{\bar{G}}\left(\mathbb{P}^1\right) \to D^b\left(\vec{Q}_T\right)$$

is an equivalence of triangulated categories.

Important Theorem II

3. Let T, T' be obtained from each by an elementary sink to source transformation: $T' = s_i^+(T)$. Then the following diagram is commutative:



   where $R\Phi_i^+$is the derived reflection functor.

4. $R\Psi_T$ identifies the derived Coxeter functor $R\mathbf{C}^+ : D^b\left(\vec{Q}_T\right) \to D^b\left(\vec{Q}_T\right)$ with the twist functor

$$\mathcal{F} \mapsto \mathcal{F}(-2)$$

   on $D^b_{\bar{G}}\left(\mathbb{P}^1\right)$.

Corollary

I'm not a massive fan of triangulated categories and I personally think the proof of this theorem is not enlightening on the representation theory side, so I have chosen to omit it.

Corollary ([Kir16] 8.26)

Let $K_{\bar{G}}\left(\mathbb{P}^1\right)$ *be the* K*-group of the category* $\mathrm{Coh}_{\bar{G}}\left(\mathbb{P}^1\right)$ *or, equivalently, of the category* $D^b_{\bar{G}}\left(\mathbb{P}^1\right)$*. Then a choice of a slice* $T \in \mathbb{Z}Q$ *gives an isomorphism* $\psi_T : K_{\bar{G}}\left(\mathbb{P}^1\right) \to L$*, where* L *is the root lattice of* Q*. This isomorphism has the following properties:*

1. $\psi_T(\mathcal{F}(-2)) = C\psi_T(\mathcal{F})$*, where* C *is the Coxeter element in* W*, adapted to the orientation* $\vec{Q}_T$*.*
2. *If* $q = (i, h_i) \in T$*, then* $\psi_T\left(X_q\right) = [P(i)]$*.*
3. *We have* $\langle \delta, \psi(\mathcal{F})\rangle = rk\mathcal{F}$*, where* rk *is the rank of sheaf* $\mathcal{F}$*.*

Geometric Construction of Representations

Recall that when Q is a tree, every orientation of Q can be obtained from a slice T. Thus, we now have a tool to geometrically construct representations of any Euclidean quiver that is a tree. We can immediately use this tool to find all indecomposable representations if we can classify all indecomposable equivariant sheaves on $\mathbb{P}^1$.

Classifying Indecomposable Sheaves

Recall that a coherent sheaf $\mathcal{F}$ on a variety X is called a torsion sheaf if its stalk at a generic point is zero. For example, if X is defined over $\mathbb{C}$, then for any $x \in X$ we have the skyscraper sheaf $\mathbb{C}_x$ whose stalk at x is $\mathbb{C}$ and at all other points is zero. As a module over the structure sheaf $\mathcal{O}$, it can be defined as $\mathbb{C}_x = \mathcal{O}_X / m_x$, where m_x is the ideal sheaf consisting of functions vanishing at x. More generally, for any $x \in X, n \geq 1$, we can define the sheaf

$$\mathbb{C}_{x,n} = \mathcal{O}_X / m_x^n$$

If x is a nonsingular point on a curve and t is a local coordinate at x, then the stalk of $\mathbb{C}_{x,n}$ at x is isomorphic to $\mathbb{C}[t]/t^n$.

Lemma

Lemma ([Kir16] 8.27)

Let $x \in \mathbb{P}^1$ and let $\bar{G}_x \subset \bar{G}$ be the stabilizer of x. Let Y be a finite-dimensional representation of $\bar{G}_x$. Then for any $n \geq 1$, there is a unique $\bar{G}$-equivariant sheaf $Y_{\bar{G}x,n}$ with the following properties:

1. *The support of $Y_{\bar{G}x,n}$ is the $\bar{G}$-orbit of x.*
2. *The stalk of $Y_{\bar{G}x,n}$ at x is $Y \otimes \mathcal{O}/m_x^n$ (as a representation of $\bar{G}_x$). For $n = 1$, we will use the shorter notation $Y_{\bar{G}x,1} = Y_{\bar{G}x}$.*

We also see that if $n = 1, \Gamma\left(Y_{\bar{G}x}\right)$ considered as a representation of $\bar{G}$ is the induced representation $\mathrm{Ind}_{\bar{G}_x}^{\bar{G}}(Y)$. In particular, if $x \in \mathbb{P}^1$ has a trivial stabilizer, then $\Gamma\left(\mathbb{C}_{\bar{G}x}\right)$ is the regular representation of $\bar{G}$.

Classification of Indecomposable Sheaves

Theorem ([Kir16] 8.29)

1. *The following is a full list of nonzero indecomposable objects in* $\mathrm{Coh}_{\bar{G}}\left(\mathbb{P}^1\right)$ *:*
   1.1 *Locally free sheaves* $\rho_i \otimes \mathcal{O}(n), i \in \mathrm{Irr}(G), n \in \mathbb{Z}, p(i) + n \equiv 0 \bmod 2$.
   1.2 *Torsion sheaves* $\mathbb{C}_{\bar{G}x,n}$*, where* $n > 0, x \in \mathbb{P}^1$ *is generic (i.e. has trivial stabilizer in* $\bar{G}$ *).*
   1.3 *Torsion sheaves* $Y_{\bar{G}x,n}$*, where* $n > 0, x \in \mathbb{P}^1$ *has nontrivial stabilizer* $\bar{G}_x$ *in* $\bar{G}$*, and* Y *is an irreducible representation of* $\bar{G}_x$*. (The pair* (x, Y) *is considered up to the action of* $\bar{G}$*.)*
2. *Indecomposable objects of* $D^b_{\bar{G}}\left(\mathbb{P}^1\right)$ *are of the form* $X[k]$*, where* X *is an indecomposable object of* $\mathrm{Coh}_{\bar{G}}\left(\mathbb{P}^1\right), k \in \mathbb{Z}$.

Classification of Indecomposable Representations

Combining this with the Important Theorem (8.24), we see that indecomposable objects in Rep $\vec{Q}$ must be of the form $R\Psi_T(X)[n]$, where X is an indecomposable object in $\mathcal{C}$ and n is chosen so that $R\Psi_T(X)[n] \in \text{Rep}\,\vec{Q}$. By Lemma 8.21, for $p \succcurlyeq T$, we have $R^1\Psi_T(X_p) = 0$, so $R\Psi_T(X_p) = \Psi(X_p) \in \text{Rep}\,\vec{Q}$. Similarly, if $p \prec T$, then $\Psi_T(X_p) = 0$, and $R\Psi_T(X_p) = R^1\Psi_T(X_p)[-1] \in \text{Rep}\,\vec{Q}[-1]$, so $R\Psi_T(X_p)[1] \in \text{Rep}\,\vec{Q}$. For an indecomposable torsion sheaf X, it is easy to check using Serre duality that $\text{Ext}^1(\mathcal{F}, X) = 0$ for any locally free sheaf $\mathcal{F}$, so $R^1\Psi_T(X) = 0$. Thus, $R\Psi_T(\mathcal{F}) = \Psi(\mathcal{F}) \in \text{Rep}\,\vec{Q}$.

Classification of Indecomposable Representations II

So, we arrive at the following result:

Theorem ([Kir16] 8.30)

In the assumptions of the Important Theorem (8.24), the following is a full list of indecomposable objects in $\operatorname{Rep}\vec{Q}_T$ *:*

1. $\Psi_T\left(\rho_i \otimes \mathcal{O}(n)\right), (i, n) \succcurlyeq T$. *These objects are preprojective.*
2. $R^1\Psi_T\left(\rho_i \otimes \mathcal{O}(n)\right) = R\Psi_T\left(\rho_i \otimes \mathcal{O}(n)\right)[1]$, $(i, n) \prec T$. *These objects are preinjective.*
3. $\Psi_T(X)$, *where* X *is an indecomposable* $\bar{G}$*-equivariant torsion sheaf on* $\mathbb{P}^1$ *described in Theorem 8.29. These objects are regular.*

Example

Example

Let $G = \{\pm I\}, \bar{G} = \{1\}$. Then the Important Theorem shows that we have an equivalence of derived categories

$$D^b\left(\operatorname{Coh}\mathbb{P}^1\right) \simeq D^b(\operatorname{Rep} K),$$

where K is the Kronecker quiver.

Conclusion

Thank you!

References

[Kir16] Alexander Jr Kirillov. *Quiver Representations and Quiver Varieties*. American Mathematical Society, 2016.