McKay Correspondence I

Max Steinberg

University of California, Los Angeles

Setup

Recall that SU(2) is the universal cover of SO(3). In particular, $\pi : \mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$ is surjective with kernel $\{\pm I\}$. We can embed cyclic groups into SU(2), eg.

$$\phi : C_n \to \mathrm{SU}(2), k \mapsto \begin{bmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{bmatrix}$$

where ζ is a primitive n-th root of unity. Thus, all cyclic groups are finite subgroups of SU(2). If n is even, then $-I \in G$, so $\pi^{-1}(\pi(G)) = G$ and $\pi(G) \cong C_{n/2}$. If n is odd, $-I \notin G$, so G cannot be written as $\pi^{-1}(\pi(G))$, and $\pi(G) = G = C_n$.

More Finite Subgroups

In general, we can find finite subgroups of SO(3) by considering symmetries of spherical polyhedra. Since SO(3) naturally acts via rotations on S^2, it is natural to consider polyhedra, since their reflectional symmetries are finite subgroups of the group of spherical rotations, SO(3).

So let X be a convex regular polyhedron in $\mathbb{R}^3$. It is well known that there are five regular convex polyhedra, called Platonic solids: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. We can describe these polyhedra via a *Schläfli Symbol* $\{p, q\}$, where p is the number of sides on each face and q is the number of faces that meet at each vertex (so, eg., a cube is $\{4, 3\}$).

Converting to the Sphere

Given a convex polyhedron X, we can project X onto the sphere S^2 to obtain a *spherical polyhedron*. We call a spherical polyhedron *regular* if it is the projection of a regular polyhedron. We also allow $\{p, 2\}$ to be regular: this is the spherical polyhedron given by two p-gons attached at every edge, each of which covers one hemisphere.

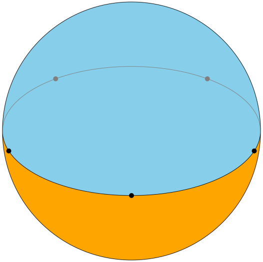

Figure: $\{5, 2\}$

Symmetries

Now, let Y be a spherical polyhedron. Clearly the group $\mathrm{Sym}(Y)$ is a finite group, and since Y is embedded into S^2, $\mathrm{Sym}(Y) \subset \mathrm{SO}(3)$. So let $G = \pi^{-1}(\mathrm{Sym}(Y))$. Finite subgroups of SU(2) of this form are called *binary polyhedral* groups. We can classify the binary polyhedral groups:

Polyhedron	$\{p, q\}$	G	$\lvert G \rvert$
n-gon, $n \geq 2$	$\{n, 2\}$	binary dihedral group BD_{4n}	$4n$
tetrahedron	$\{3, 3\}$	binary tetrahedral	24
octahedron cube	$\{3, 4\}$ $\{4, 3\}$	binary octahedral	48
isocahedron dodecahedron	$\{3, 5\}$ $\{5, 3\}$	binary icosahedral	120

Figure: [Kir16] 135

This can actually be proven using quaternions as we will briefly touch on later.

Classification of Finite Subgroups of SU(2)

Theorem ([Kir16] 8.2)

Every nontrivial finite subgroup of $\mathrm{SU}(2)$ *is isomorphic to a cyclic group or a binary polyhedral group.*

Corollary ([Kir16] 8.3)

Every finite subgroup of $\mathrm{SU}(2)$ *is isomorphic to* $\pi^{-1}(\overline{G})$ *for some* $\overline{G} \subset \mathrm{SO}(3)$*, or isomorphic to* C_n *for* n *odd.*

Barycentric Subdivision

Let X be a regular spherical polyhedron and G be its corresponding binary polyhedral group. We can perform barycentric subdivision on X (example below for the binary icosahedral group) and obtain a triangulation of S^2 that is two-colourable. Each triangle in this triangulation has angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{2}$ (notably this is invariant under $p \leftrightarrow q$, as we would expect since dual polyhedra have the same symmetries).

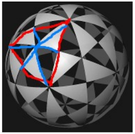

Figure: [Kir16] 136, edited by me (that's why it looks awful)

Classification, I

Theorem ([Kir16] 8.4)

Let X be a $\{p,q\}$ regular spherical polyhedron and $G \subset \mathrm{SU}(2)$ be the corresponding binary polyhedral group (and $\overline{G} = \pi(G)$). We consider the spherical triangulation detailed previously.

1. *The union of two adjacent spherical triangles is a fundamental domain for the action of $\overline{G}$ on S^2.*
2. *There are precisely three $\overline{G}$-orbits in $\mathbb{P}^1 \cong S^2$ which have nontrivial stabiliser:*
   - 2.1 *Centres of faces of X – stabilisers in G are C_{2p}*
   - 2.2 *Vertices of X – stabilisers in G are C_{2q}*
   - 2.3 *Midpoints of edges of X – stabilisers in G are C_4*

Classification, II

Continued from the previous slide.

Theorem ([Kir16] 8.4)

3. *Let Δ be a triangle in the barycentric subdivision of X with vertices $(v_0, v_1, v_2) \in \mathbb{P}^1$ so that v_0 is a vertex of X, v_1 is a midpoint of an edge incident to v_0, and v_2 is the centre of a face incident to that edge. Let A, B, C be generators of the stabilisers $G_{v_2}, G_{v_0}, G_{v_1}$ respectively so that $\pi(A)$ is the clockwise rotation by $\frac{2\pi}{p}$ about v_2, similar for B, C. Then $A^p = B^q = C^2 = -I$, and G is generated by A, B, C.*

This is an extremely cumbersome definition that actually offers a much simpler depiction via quaternions.

Example

Let BD_{4n} denote the binary dihedral group $BD_{4n} = \pi^{-1}(D_{2n})$. This has Schläfli symbol $\{p, 2\}$. The orbits in $\mathbb{P}^1$ with nontrivial stabiliser are:

1. $\overline{G} \cdot 0 = \{0, \infty\}$, corresponding to the centres of faces of X – each stabiliser is isomorphic to C_{2n}
2. $\overline{G} \cdot 1$ is an orbit of order n, corresponding to the vertices of X – each stabiliser is isomorphic to C_4
3. $\overline{G} \cdot \zeta$ is an orbit of order n, corresponding to the midpoints of edges of X – each stabiliser is isomorphic to C_4

where ζ is the choice of principal $2n$-th root of unity used to embed D_{2n} into SO(3).

Presentation of G

Theorem ([Kir16] 8.6)

Let $G \subset \mathrm{SU}(2)$ be a finite subgroup. If $G = C_n$, pick any $p, q > 0$ so that $p + q = n$ and set $r = 1$. If G is a binary polyhedral group, set p, q so that G has Schläfli symbol $\{p, q\}$ and $r = 2$. Then,

1. $G \cong \langle A, B, C | A^p = B^q = C^r = ABC \rangle$.
2. $A^i, B^j, C^k, 1 \leq i \leq p-1, 1 \leq j \leq q-1, 1 \leq k \leq r-1$ *are representatives for the nonidentity conjugacy classes of G.*

Corollary

Corollary

$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$

Proof.

We have a spherical triangle S with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$. The area of such a triangle is given by $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} - \pi > 0$, so dividing through by π shows $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 > 0$. □

Define $\Gamma(p, q, r)$ to be the *star quiver*: it is constructed by three branches, each with length p, q, r, meeting at a single point. We count the joining point in the length of each branch, so when $r = 1$, we have only two branches. Very early on, we proved that if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$, then $\Gamma(p, q, r)$ is Dynkin.

Example

Thus, we can directly map finite subgroups $G \subset \mathrm{SU}(2)$ to Dynkin graphs. We can prove this is a bijection by appealing to classifications of both sets of objects. In addition, we can label each vertex in the following way: in the length p branch, we label the vertices $A, A^2, \ldots, A^{p-1}$, and similar for the other two branches.

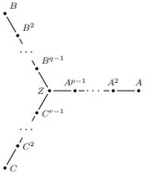

Figure: [Kir16] 140

ADE Classification

Theorem ([Kir16] 8.9)

We have a bijection between nontrivial finite subgroups $G \subset \mathrm{SU}(2)$ up to conjugation and Dynkin graphs. For every finite subgroup G, the vertices of the corresponding Dynkin graph are in bijection with nonidentity conjugacy classes in G.

G	(p,q,r)	Γ
$\mathbb{Z}_n$, $n \geq 2$	$(p,q,1)$, $p+q=n$	A_{n-1}
binary dihedral group BD_{4n}, $n \geq 2$	$(n,2,2)$	D_{n+2}
binary tetrahedral	$(3,3,2)$	E_6
binary octahedral	$(4,3,2)$	E_7
binary icosahedral	$(5,3,2)$	E_8

Figure: [Kir16] 140

Quaternions

Let $G \leq \mathbb{H}$ be a finite multiplicative subgroup of $\mathbb{H}$. Since norm is multiplicative, $|g| = 1 \forall g \in G$ since G is finite.
Let $\overline{\mathbb{H}} \subset \mathbb{H}$ denote the unit-norm quaternions, so that $\mathrm{SU}(2) \cong \overline{\mathbb{H}}$. Then clearly every finite subgroup of SU(2) bijectively corresponds to a finite subgroup of $\overline{\mathbb{H}}$ and by the above argument, bijectively corresponds to a finite (multiplicative) subgroup of $\mathbb{H}$.

Finite Subgroups of Quaternions

In [Cox74], Coxeter provides a classification of all finite multiplicative subgroups of $\mathbb{H}$ using the same geometric tools we have used, but in the context of quaternions. In particular,

$$G(p, q, r) \cong \langle A, B, C | A^p = B^q = C^r = ABC \rangle$$

is generated as a multiplicative group of quaternions by

$$A = \exp(P \cdot \pi/p), B = \exp(Q \cdot \pi/q), C = \exp(R \cdot \pi/r)$$

where P, Q, R are the (unit quaternion) points on the spherical triangles in the barycentric subdivision (with angle π/p at P, π/q at Q, and π/r at R). There is also a connection between the Cayley graph of G and the McKay quiver of G (to be defined momentarily).

Setup

Let G be a finite subgroup of $\mathrm{SU}(2)$ and $I = \mathrm{Irr}(G)$ be the set of isomorphism classes of irreducible representations of G. We denote ρ_i to be an arbitrary irreducible element of the class i for every $i \in I$. In particular, $\rho_0 = \mathbb{C}$, the trivial representation. $K(G)$ is the Grothendieck group of $\mathrm{Rep}(G)$, a free abelian group with basis $[\rho_i], i \in I$. Under the tensor product, it becomes a commutative ring. It also has a natural symmetric bilinear form $([X], [Y])_0 = \dim \mathrm{Hom}_G(X, Y)$, satisfying $([\rho_i], [\rho_j])_0 = \delta_{ij}$.

Building a Quiver, I

Let ρ be the natural representation of SU(2) on $\mathbb{C}^2$ restricted to G. Since SU(2) is clearly unitary, $\rho^* = \rho$. So let $A : K(G) \to K(G)$ be given by $A(x) = [\rho] \cdot x$. In the $[\rho_i]$ basis, A can be represented by a matrix: $A_{ij} = \dim \mathrm{Hom}_G(\rho_i, \rho \otimes \rho_j)$. Since $\rho^* = \rho$, A is symmetric.

Now, define the bilinear form on $K(G) \otimes_{\mathbb{Z}} \mathbb{R}$ given by $(x, y) = (x, (2 - A)y)_0$. This is symmetric and positive semidefinite, and has a radical generated by $\delta = \sum \dim(\rho_i)[\rho_i] \in K(G)$.

Building a Quiver, II

Theorem ([Kir16] 8.13)

Let G be a nontrivial finite subgroup of $\mathrm{SU}(2)$ and $Q(G)$ be a finite graph with vertices $I = \mathrm{Irr}(G)$, and number of edges between i and j given by A_{ij}. Then Q is a connected Euclidean graph, ρ_0 is the extending vertex, and the class of the regular representation in $K(G)$ is the generator of the radical of the Euler form.

Proof of Theorem

Proof.

The root lattice $L(Q)$ is precisely $K(G)$ and $(-,-)_Q$ is precisely $(-,-)$ we just defined, so it is immediately positive semidefinite. To see Q is connected, we recall that every irreducible representation of G is a subrepresentation of $\rho^{\otimes n}$ for some $n \gg 0$ (since ρ is clearly faithful). The number of edges connecting i and j is the multiplicity of ρ_i in $\rho_j \otimes \rho^{\otimes n}$ for some n and we are done. □

McKay's Theorem

We now arrive at the key point of this talk: McKay's Theorem.

Theorem (McKay)

The previous construction defines a bijection between nontrivial finite subgroups $G \subset \mathrm{SU}(2)$ up to conjugation and connected Euclidean graphs up to isomorphism other than the Jordan graph.

McKay Correspondence is traditionally extended via the correspondence $\{e\} \leftrightarrow J$, mapping the trivial subgroup to the Jordan quiver.

Example

Let $G = C_n$ embedded in SU(2) as we have previously done. G has n irreducible representations, all of dimension 1 (it is abelian). The tensor product is given by $\rho_i \otimes \rho_j = \rho_{i+j}$ (such that $i + j$ is taken modulo n), and $\rho \cong \rho_1 \oplus \rho_{-1}$. Thus, $A[\rho_i] = [\rho_{i+1}] + [\rho_{i-1}]$. This is precisely the incidence matrix of $\hat{A}_{n-1}$, the Euclidean graph of type A_{n-1} with n vertices.

G	Q	$\lvert I\rvert$
$\{1\}$	J	1
$\mathbb{Z}_n$, $n \geq 2$	$\widehat{A}_{n-1}$	n
binary dihedral group BD_{4n}, $n \geq 2$	$\widehat{D}_{n+2}$	$n+3$
binary tetrahedral	$\widehat{E}_6$	7
binary octahedral	$\widehat{E}_7$	8
binary icosahedral	$\widehat{E}_8$	9

Figure: [Kir16] 144, full classification of McKay Correspondence

Proof of McKay's Theorem?

This theorem can be proven explicitly by simply classifying Euclidean quivers and finite subgroups of SU(2) (as we have done), but this is not conceptually interesting. In the case of Dynkin graphs, we had a similar construction that appeared to be proven via classification rather than conceptual understanding. We will see much later in the summer that these constructions are related through the language of Kleinian singularities.
Just an interesting note, Dynkin diagrammes of simply-laced affine (untwisted) Lie algebras are precisely the connected Euclidean graphs (other than the Jordan quiver) and in this case we have the Cartan matrix $C = 2I - A$.

Can we generalise this?

A natural question might be, "is $\mathbb{H}$ the only option?" Going to smaller $\mathbb{R}$-algebras, $\mathbb{R}$ clearly has no interesting finite multiplicative subgroups, and $\mathbb{C}$ has only cyclic groups. What if we go to *larger* $\mathbb{R}$-algebras?

Octonions!

It turns out that there is actually something going on with octonions. Considering finite subgroups of $G_2 := \mathrm{Aut}_{\mathbb{R}}(\mathbb{O})$ (just as $\mathrm{SU}(2) \cong \overline{\mathbb{H}} = \mathrm{Aut}_{\mathbb{R}}(\mathbb{H})$), there is some sort of correspondence going on. Clearly any finite multiplicative subgroup of $\mathbb{H}$ is also a multiplicative subgroup of G_2, but as far as I am aware it is currently unknown what combinatorial relation McKay quivers of finite subgroups of G_2 satisfy.

Conclusion

Thank you!

References

[Cox74] H. Coxeter. *Regular Complex Polytopes.* 1974.

[Kir16] Alexander Jr Kirillov. *Quiver Representations and Quiver Varieties.* American Mathematical Society, 2016.