Limits, Colimits, and More Abstract ∞-Nonsense

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1 Introduction

On Friday, Zach introduced us to ∞-adjunctions and some of the formalisms behind them. Now, we will investigate one of the most useful and important applications of limits: defining limits and colimits in the ∞-setting. Specifically, we want to be able to take (co)limits valued *in* an ∞-category A in some ∞-cosmos K.

1.1 Definitions, First Time

A priori, a lot of what we are about to do looks like we are just writing ∞ in front of a lot of traditional 1-category theory defintions. In a certain sense we are, and we will later see that this definition does not generalise exactly how we want it to, so we will have to refine it later. The true idea of ∞-(co)limits in an ∞-cosmos is a lot more subtle and requires some clever techniques to define properly. We will start with the "basic" definitions that lead us to our first definition of a limit.

Definition 1. *(diagram ∞-category)*

1. *Let K be an ∞-cosmos and $A \in K$ be an ∞-category. Let J be a simplicial set. Then A^J is the* ***∞-category of J-shaped diagrams in*** *A.*

2. *Let K' be a cartesian closed ∞-cosmos and $A, J \in K'$ be ∞-categories. Then A^J is the* ***∞-category of J-shaped diagrams in*** *A, and there is a natural bijection between A^J and functors $d : J \to A$.*

We call any element $d : 1 \to A^J$ a ***diagram of shape J in*** *A (or simply a* ***diagram*** *when J and A are clear).*

Quick remark: every ∞-topos is a cartesian closed ∞-cosmos (don't ask about this because I don't know anything about topos theory), and our second definition really amounts to saying A^J is an internal hom in a cartesian closed ∞-cosmos. Both of these definitions induce simplicial bifunctors

$$\mathsf{SSet} \times K \to K, (J, A) \mapsto A^J$$

$$K \times K \to K, (J, A) \mapsto A^J$$

The difference between these two types of diagrams is rather technical but in most standard contexts these are the same (eg. in the ∞-cosmos of quasicategories these are immediately identical and in most cases a diagram indexed by an ∞-category J is equivalent to the diagram indexed by the quasicategory inside of J, regarded as a simplicial set). For the rest of this talk a diagram will be a simplicial set indexed diagram unless otherwise specified. All results can be transfered to the ∞-category indexed case.

Definition 2. *(constant diagram functor) There is a terminal object* 1 *satisfying* $A^1 \cong A$ *for any* ∞*-category* A*. Restriction along the unique map* $! : J \to 1$ *induces the* ***constant diagram functor*** $\Delta : A \to A^J$*.*

You may recall from traditional 1-category theory that the limit and colimit are adjoints to the constant diagram functor. This leads us to our first definition of ∞-(co)limits:

Definition 3. *(admitting all (co)limits) Let* A *be an* ∞*-category and* J *be a simplicial set.*

1. *A* ***admits all limits of shape*** *J if $\Delta : A \to A^J$ admits a right adjoint.*
2. *A* ***admits all colimits of shape*** *J if $\Delta : A \to A^J$ admits a left adjoint.*

$$\begin{tikzcd}
A^J \arrow[d, bend right=70, "\mathrm{colim}"'] \arrow[d, bend left=70, "\lim"] \\
A \arrow[u, "\Delta" description]
\end{tikzcd}$$

(Using Balmer's notation for adjunctions). In the ∞-cosmos of categories, these definitions are just the usual limits and colimits from traditional 1-category theory as we mentioned previously.

Definition 4. *((co)products) Let* J *be a set and* $\coprod_J 1$ *be the traditional coproduct of the terminal object* 1 *indexed by* J*. Then a limit by diagram* J *is called a* ***product*** *and a colimit is called a* ***coproduct****.*

Lemma 1. *Given an* ∞*-category* A*, products and coproducts in* A *also define products and coproducts in* $h(A)$*.*

Proof. We have $A^J \cong \prod_J A$ (as an equivalence of ∞-categories) when J is a set. (We can think of this as $\mathrm{Hom}(\coprod 1, A) \cong \prod \mathrm{Hom}(1, A)$ and by construction $\mathrm{Hom}(1, A) = A^1 \cong A$.) Then since the homotopy category functor preserves limits, we have

$$h(A^J) \cong h\left(\prod_J A\right) = \prod_J h(A) \cong (hA)^J$$

So our adjoints descend to the homotopy category

$$
\begin{tikzcd}
(hA)^J \arrow[d, phantom, "\cong"] \\
h(A^J) \arrow[d, bend right=70, "\mathrm{colim}"'] \arrow[d, bend left=70, "\lim"] \\
hA \arrow[u, "\Delta"]
\end{tikzcd}
$$

□

1.2 That Was a Bad Definition

Philosophically, it is a little bit dangerous to get overly attached to the homotopy category when dealing with these constructions. While the relation we just detailed holds for (co)products, it does not hold for general (co)limits. For example, $A^{\Delta[1]}$ represents the 1-simplices in the underlying quasicategory of A, and certain cotensors do not descend to the homotopy category properly.

An issue with previous definition on the grounds that the existence of a certain (co)limit is governed completely by the diagram shape and not the actual diagram. In traditional 1-category theory, we can see this very often: take for example a category that does not have all equalisers but has some. Then the existence of $\lim(X \rightrightarrows Y)$ depends on the choice of X and Y, but with the previous definition, it never exists! So we would like to generalise our definition from "all (co)limits of this shape exist" to "the (co)limit of this diagram exists".

1.3 Definitions, Second Time

In order to define (co)limits of diagrams, we need absolute Kan extensions. You may have seen this in the traditional 1-categorical setting.

Definition 5. *(absolute left lifting) Let $C \xrightarrow{g} A \xleftarrow{f} B$ be a cospan in a 2-category. Then an **absolute left lifting** of g through f is the data of a 1-cell ℓ and a 2-cell λ:*

$$
\begin{tikzcd}
 & B \arrow[d, "f"] \\
C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"'] & A
\end{tikzcd}
$$

With the following universal property: every 2-cell on the left factors uniquely (shown below-right):

$$
\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Uparrow\chi"] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\exists!\Uparrow\zeta", near start] & B \arrow[d, "f"] \\
C \arrow[ur, "\ell"] \arrow[r, "g"'] & A \arrow[ul, phantom, "\Uparrow\lambda", near start]
\end{tikzcd}
$$

Dually,

Definition 6. *(absolute right lifting) Let $C \xrightarrow{g} A \xleftarrow{f} B$ be a cospan in a 2-category. Then an **absolute right lifting** of g through f is the data of a 1-cell r and a 2-cell ρ:*

$$
\begin{tikzcd}
& B \arrow[d, "f"] \\
C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[r, phantom, "\Downarrow\rho", shift left=3ex] & A
\end{tikzcd}
$$

With the following universal property: every 2-cell on the left factors uniquely (shown below-right):

$$
\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\exists!\Downarrow\zeta", near start] & B \arrow[d, "f"] \\
C \arrow[ur, "r"] \arrow[r, "g"'] & A \arrow[ul, phantom, "\Downarrow\rho", near start]
\end{tikzcd}
$$

For simplicity (and diverging from the source text), we call a diagram a **absolute lifting diagram** if it is either an absolute left lifting or an absolute right lifting. Let's explore some facts and examples of absolute lifing diagrams.

Lemma 2. *(Restriction lemma) Absolute lifting diagrams are invariant under restriction of domain: if (ℓ, λ) is an absolute left lifting of g through f, then $c : X \to C$ gives $(\ell c, \lambda c)$, a lifting of gc through f (and dually for right liftings).*

Lemma 3. *$\eta : id_B \Rightarrow uf$ is the unit of $f \dashv u$ iff (f, η) is an absolute left lifting of the identity through u. Dually, $\epsilon : fu \Rightarrow id_A$ is the counit of $f \dashv u$ iff (u, ϵ) is an absolute right lifting of the identity through f.*

$$
\begin{tikzcd}
& A \arrow[d, "u"] \\
B \arrow[ur, "f"] \arrow[r, equal] \arrow[r, phantom, "\Uparrow\eta", shift left=3ex] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& B \arrow[d, "f"] \\
A \arrow[ur, "u"] \arrow[r, equal] \arrow[r, phantom, "\Downarrow\epsilon", shift left=3ex] & A
\end{tikzcd}
$$

Recall that our first definition gave us colim $\dashv \Delta \dashv$ lim. We would like to use our new technology to define the (co)limit of an individual diagram.

$$
\begin{tikzcd}
& A \arrow[d, "\Delta"] \\
A^J \arrow[ur, "\mathrm{colim}"] \arrow[r, equal] \arrow[r, phantom, "\Uparrow\eta", shift left=3ex] & A^J
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[d, "\Delta"] \\
A^J \arrow[ur, "\lim"] \arrow[r, equal] \arrow[r, phantom, "\Downarrow\epsilon", shift left=3ex] & A^J
\end{tikzcd}
$$

By the restriction lemma, we can restrict to any subobject of the ∞-category of diagrams and maintain the same universality. As a result, we can define the (co)limit of any family of diagrams:

Definition 7. *(limit and colimit) Let $d : D \to A^J$. We call d a **family of diagrams of shape** J. Then the **colimit** of d is an absolute left lifting*

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\operatorname{colim} d"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3, "\Uparrow\eta"] & A^J \end{tikzcd}$$

*where $\operatorname{colim} d : D \to A$ is a generalised element and $\eta : d \Rightarrow \Delta \operatorname{colim} d$ is a **colimit cone**. Dually, the **limit** of d is an absolute right lifting diagram*

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim d"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3, "\Downarrow\epsilon"] & A^J \end{tikzcd}$$

*where $\lim d : D \to A$ is a generalised element and $\epsilon : \Delta \lim d \Rightarrow d$ is a **limit cone**.*

As you might expect, when A has all limits of shape J, then any family of diagrams $d : D \to A^J$ has a limit. It is not always true that if every diagram $d : 1 \to A^J$ has a limit that A has all limits of shape J. As an example, as in traditional 1-category theory, an initial element is the colimit of an empty diagram.

$$\begin{tikzcd} & A \arrow[d, "!"] \\ 1 \arrow[ur, "i"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\parallel"] & 1 \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Uparrow\chi"] & A \arrow[d, "!"] \\ 1 \arrow[r, equal] & 1 \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, shift left=3, "\exists!\Uparrow\zeta"] & A \arrow[d, "!"] \\ 1 \arrow[ur, "i"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\parallel"] & 1 \end{tikzcd}$$

In the essence of time I won't talk about split augmented (co)simplicial objects and geometric realisation, but if there is time I may say a bit about them at the end.

2 Preservation of Limits and Colimits

2.1 Introduction

Recall in traditional 1-category theory, we say a functor $f : A \to B$

1. *preserves* limits if it sends limit cones in A to limit cones in B
2. *reflects* limits if cones in A that are sent to limit cones in B are limit cones in A

3. *creates* limits if whenever a diagram in A admits a limit in B, there is a limit cone in A that maps to a cone isomorphic to the limit cone in B

We will use these definitions verbatim for functors of ∞-categories. One of the most famous results from traditional 1-category theory, and one that I've used many times in 210 homeworks and exams, is that left adjoints preserve colimits and right adjoints preserve limits. We will prove this in the ∞-category context very shortly.

2.2 Adjoints and Limits

First, we need a lemma about absolute lifting diagrams.

Lemma 4. *(composition and cancellation of absolute lifting diagrams) Consider the following diagram where (r, ρ) is an absolute right lifting of h through f:*

$$\begin{tikzcd}
& C \arrow[d, "g"] \\
& B \arrow[d, "f"] \\
D \arrow[r, "h"'] \arrow[ur, "r"] \arrow[uur, bend left, "s"] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] \arrow[uur, phantom, shift right=2, "\Downarrow\sigma"] & A
\end{tikzcd}$$

Then (s, σ) is a right lifting of r through g iff $(s, \rho \cdot f\sigma)$ is an absolute right lifting of h through fg.

Now, we can prove right adjoints preserve limits. Left adjoints preserving colimits is simply the dual of the following proof.

Proposition 1. *Right adjoints preserve limits.*

Proof. (Proof sketch, the full proof is given in the text). Let K be an ∞-cosmos, $A, B \in K$ be ∞-categories, and $u : A \to B$ a functor with a left adjoint $f : B \to A$ and counit $\epsilon : fu \Rightarrow id_A$. We want to show that absolute right liftings below-left are sent to absolute right liftings below-right.

$$\begin{tikzcd}
& A \arrow[d, "\Delta"] \\
D \arrow[r, "d"'] \arrow[ur, "\lim d"] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A^J
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& A \arrow[d, "\Delta"] \arrow[r, "u"] & B \arrow[d, "\Delta"] \\
D \arrow[r, "d"'] \arrow[ur, "\lim d"] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A^J \arrow[r, "u^J"'] & B^J
\end{tikzcd}
\tag{2.4.3}$$

$(-)^J$ sends adjunctions to adjunctions and counits to counits, so $f^J \dashv u^J$ with counit ϵ^J. Then by the previous lemma, the above-right diagram is an absolute right lifting if and only if the below-left diagram is. We can paste the 2-cell with the counit, which transposes the cone across the adjunction $f^J \dashv u^J$, and we arrive at the below-center diagram. Then contracting across $A = A, A^J = A^J$, we arrive at a diagram below-right which is an absolute right lifting by the previous lemma.

$$
\begin{tikzcd}
& A \arrow[r, "u"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\
D \arrow[ur, "\lim d"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A^J \arrow[r, "u^J"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\Downarrow\epsilon^J"] & B^J \arrow[d, "f^J"] \\
& & A^J
\end{tikzcd}
=
\begin{tikzcd}
& & B \arrow[d, "f"] \\
& A \arrow[ur, "u"] \arrow[r, equal] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon"] \arrow[d, "\Delta"] & A \arrow[d, "\Delta"] \\
D \arrow[ur, "\lim d"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A^J \arrow[r, equal] & A^J
\end{tikzcd}
=
\begin{tikzcd}
& B \arrow[d, "f"] \\
& A \arrow[d, "\Delta"] \\
D \arrow[uur, bend left, "u \lim d"] \arrow[uur, phantom, "\Downarrow\epsilon \lim d"] \arrow[ur, "\lim d"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A^J
\end{tikzcd}
$$

□

Proposition 2. *An equivalence of ∞-categories preserves, reflects, and creates limits and colimits.*

Definition 8. *(initial and final functors) A functor $k : I \to J$ is **final** if a J-shaped cone defines a colimit cone if and only if the restricted I-shaped cone is a colimit cone, and **initial** if a J-shaped cone defines a limit cone if and only if the restricted I-shaped cone is a limit cone.*

Proposition 3. *Left adjoints are initial and right adjoints are final.*

Proposition 4. *Fully faithful functors $f : A \to B$ reflect any limits or colimits that exist in B.*

Proofs of all of these propositions are available in the text but I've chosen not to go through them for time.